13 b) NEURČITÝ INTEGRÁL

D: Jsou dány funkce F, f definované v otevřeném intervalu I. Jestliže pro každé $x \in I$ platí

$$F'(x) = f(x),$$

říkáme, že funkce F je primitivní funkcí k funkci f v intervalu I.

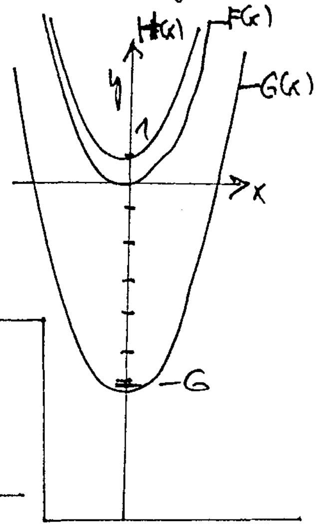
Početní postup, při kterém k dané funkci f určíme primitivní funkci F, se nazývá integrace, či integrování, popřípadě nalezení neurčitého integrálu.

Příklad 1 (1/135 - uč.): Najděte primitivní funkci k funkci $f(x) = 4x^3 + 2x$ v intervalu $(-\infty, +\infty)$.

Řešení: najít primitivní funkci k původní funkci, jejíž derivace se rovná $4x^3 + 2x$. Je to funkce $x^4 + x^2$. Ovšem platí:

$f(x) = 4x^3 + 2x$

- $x^4 + x^2$ ... $F(x)$
- $x^4 + x^2 - 6$ ... $G(x)$
- $x^4 + x^2 + 1$ ... $H(x)$ atd.



$$\int (4x^3 + 2x)\,dx = \boxed{x^4 + x^2 + C, \text{ kde } C \in R}$$

ZÁKLADNÍ VZORCE PRO PRIMITIVNÍ FUNKCE

1. $\int 0\,dx = C, \quad x \in R$

2. $\int dx = \int 1\,dx = x + C, \quad x \in R$ | $\int c\,dx = cx \quad x \in (-\infty, +\infty)$

3. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C, \quad x \in (0, +\infty), \; n \in R - \{-1\}$

Pro některá n je možné platnost tohoto vztahu rozšířit. Např. pro $n \in N$ platí pro $x \in (-\infty, +\infty)$.

4. $\int \frac{1}{x}\,dx = \ln x + C, \quad x \in (0, +\infty)$ | 5. $\int \frac{1}{x}\,dx = \ln(-x) + C, \quad x \in (-\infty, 0)$

$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad x \in (-\infty, 0), \text{ nebo } x \in (0, +\infty)$$

$\boxed{\ln 1 = 0 \mid \ln e = 1}$

$\boxed{6}$ $\int e^x\,dx = e^x + C, \quad x \in (-\infty, +\infty)$

$\boxed{7}$ $\int a^x\,dx = \dfrac{a^x}{\ln a} + C, \quad a>0,\ a \neq 1,\ x \in (-\infty, +\infty)$

$\boxed{8}$ $\int \sin x\,dx = -\cos x + C, \quad x \in (-\infty, +\infty)$

$\boxed{9}$ $\int \cos x\,dx = \sin x + C, \quad x \in (-\infty, +\infty)$

$\boxed{10}$ $\int \dfrac{1}{\cos^2 x}\,dx = \mathrm{tg}\,x + C, \quad x \in \left(-\frac{1}{2}\pi + k\pi, \frac{1}{2}\pi + k\pi\right) \quad k \in Z$

$\boxed{11}$ $\int \dfrac{1}{\sin^2 x}\,dx = -\mathrm{cotg}\,x + C, \quad x \in (k\pi, \pi + k\pi), \quad k \in Z$

<u>Věta 1:</u> $\int [c_1 f_1(x) + c_2 f_2(x)]\,dx = c_1 \int f_1(x)\,dx + c_2 \int f_2(x)\,dx$

z této věty plynou vztahy:

1. $\int c f(x)\,dx = c \int f(x)\,dx$
2. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
3. $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

<u>Příklad 2:</u> Vypočítejte s využitím vzorce $\boxed{3}$ bez použití znaku $\int$.

a) $f(x) = x^4, \ldots F(x) = \dfrac{x^{4+1}}{4+1} = \dfrac{x^5}{5} = \dfrac{1}{5}x^5, \quad x \in R \quad \ldots \boxed{\dfrac{x^5}{5} + C}, \quad x \in R$

b) $f(x) = x^{-5} \ldots F(x) = \dfrac{x^{-5+1}}{-5+1} = \dfrac{x^{-4}}{-4} = -\dfrac{1}{4x^4}, \quad x \neq 0 \quad \ldots \boxed{-\dfrac{1}{4x^4} + C} \quad x \in R$

c) $f(x) = \sqrt[3]{x} \ldots f(x) = x^{\frac{1}{3}} = \dfrac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} = \dfrac{x^{\frac{4}{3}}}{\frac{4}{3}} = \dfrac{3}{4}x^{\frac{4}{3}} =$

$= \dfrac{3}{4}\sqrt[3]{x^4} = \dfrac{3}{4}\cdot\sqrt[3]{x^3 \cdot x} = \boxed{\dfrac{3}{4}x\cdot\sqrt[3]{x} + C, \quad x \geq 0}$

d) $x^{-\frac{9}{4}} \ldots F(x) = \dfrac{x^{-\frac{9}{4}+1}}{-\frac{9}{4}+1} = \dfrac{x^{-\frac{5}{4}}}{-\frac{5}{4}} = -\dfrac{4}{5}\cdot x^{-\frac{5}{4}} = -\dfrac{4}{5}\cdot\dfrac{1}{x^{\frac{5}{4}}} =$

$= -\dfrac{4}{5}\cdot\dfrac{1}{\sqrt[4]{x^5}} = -\dfrac{4}{5}\cdot\dfrac{1}{x\cdot\sqrt[4]{x}} = -\dfrac{4}{5x\sqrt[4]{x}} = \boxed{-\dfrac{4}{5x\sqrt[4]{x}} + C, \quad x > 0}$

Příklad 3: Určete primitivní funkce, respektive neurčitý integrále s využitím vztahu $F(x) = \int f(x)\,dx$.

a) $$\int x\cdot\sqrt[3]{x^2}\,dx = \int\sqrt[3]{x^3\cdot x^2}\,dx = \int\sqrt[3]{x^5}\,dx = \int x^{\frac{5}{3}}\,dx = \frac{x^{\frac{8}{3}}}{\frac{8}{3}} =$$
$$= \frac{3}{8}\cdot\sqrt[3]{x^8} = \frac{3}{8}\sqrt[3]{x^6\cdot x^2} = \frac{3}{8}x^2\sqrt[3]{x^2} = \boxed{\frac{3}{8}x^2\sqrt[3]{x^2},\quad x\geqq 0}$$

b) $$\int\frac{x^3\cdot\sqrt{x}}{\sqrt[4]{x^3}}\,dx = \int\frac{\sqrt{x^6\cdot x}}{x^{\frac{3}{4}}}\,dx = \int\frac{\sqrt{x^7}}{x^{\frac{3}{4}}}\,dx = \int\frac{x^{\frac{7}{2}}}{x^{\frac{3}{4}}}\,dx =$$
$$= \int x^{\frac{7}{2}-\frac{3}{4}}\,dx = \int x^{\frac{11}{4}}\,dx = \frac{x^{\frac{11}{4}+1}}{\frac{11}{4}+1} = \frac{x^{\frac{15}{4}}}{\frac{15}{4}} = \frac{4}{15}\cdot\sqrt[4]{x^{15}} =$$
$$= \frac{4}{15}\sqrt[4]{x^{12}\cdot x^3} = \frac{4}{15}\cdot x^3\cdot\sqrt[4]{x^3} = \boxed{\frac{4}{15}x^3\sqrt[4]{x^3},\quad x\geqq 0}$$

c) $$\int\frac{1}{\sqrt[5]{x^4}}\,dx = \int\frac{1}{x^{\frac{4}{5}}}\,dx = \int x^{-\frac{4}{5}}\,dx = \frac{x^{\frac{1}{5}}}{\frac{1}{5}} = \boxed{5\cdot\sqrt[5]{x}+C,\ x>0}$$

d) $$\int\sqrt{\frac{x\cdot\sqrt[3]{x}}{\sqrt[4]{x^3}}}\,dx = \int\sqrt{\frac{\sqrt[3]{x^4}}{\sqrt[4]{x^3}}}\,dx = \int\sqrt{\frac{x^{\frac{4}{3}}}{x^{\frac{3}{4}}}}\,dx = \int\left(\frac{x^{\frac{4}{3}}}{x^{\frac{3}{4}}}\right)^{\frac{1}{2}}dx =$$
$$= \int\frac{x^{\frac{2}{3}}}{x^{\frac{3}{8}}}\,dx = \int x^{\frac{2}{3}-\frac{3}{8}}\,dx = \int x^{\frac{7}{24}}\,dx = \frac{x^{\frac{31}{24}}}{\frac{31}{24}} = \boxed{\frac{24}{31}x\cdot\sqrt[24]{x^{31}}+C,\ x>0}$$

Příklad 4 (ze sbírky (maturitní minimum)): Vypočítejte:

a) $y=6$ podle $\boxed{2}$ $\boxed{\int 6\,dx = 6x+C}$ $\quad x\in(\infty,+\infty)$

b) $f(x)=x^2-3x+5$ ... podle $\boxed{3}$ 1, 2, 3 ze ch.2 platí $F(x) =$
$$= \int(x^2-3x+5)\,dx = \frac{x^{2+1}}{2+1} - 3\frac{x^{1+1}}{1+1} + 5x = \frac{x^3}{3} - 3\frac{x^2}{2} + 5x$$
$$= \boxed{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 5x + C}\quad x\in\mathbb{R}$$

c) $f: y = (\sqrt[3]{x}+1)\cdot(x-1)$ ... upravíme $(x^{\frac{1}{3}}+1)\cdot(x-1)$

$$y = x^{\frac{1}{3}+1} + x - x^{\frac{1}{3}} - 1 = x^{\frac{4}{3}} + x - x^{\frac{1}{3}} - 1$$

$$\int \left(x^{\frac{4}{3}} - x^{\frac{1}{3}} + x - 1\right) dx = \frac{x^{\frac{7}{3}}}{\frac{7}{3}} - \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{x^2}{2} - x =$$

$$= \frac{3}{7} \cdot x^{\frac{7}{3}} - \frac{3}{4} x^{\frac{4}{3}} + \frac{1}{2} x^2 - x = \frac{3}{7} \sqrt[3]{x^7} - \frac{3}{4} \cdot \sqrt[3]{x^4} + \frac{1}{2} x^2 - x =$$

$$= \frac{3}{7} \sqrt[3]{x^6 \cdot x} - \frac{3}{4} \sqrt[3]{x^3 \cdot x} + \frac{1}{2} x^2 - x = \boxed{\frac{3}{7} x^2 \sqrt[3]{x} - \frac{3}{4} x \cdot \sqrt[3]{x} + \frac{1}{2} x^2 - x + C}$$

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d) $f: y = \dfrac{x + \sqrt{x}}{x}$ ... $y = \dfrac{x}{x} + \dfrac{\sqrt{x}}{x} = 1 + x^{\frac{1}{2}} \cdot x^{-1} = 1 + x^{-\frac{1}{2}}$

$$\int \left(1 + x^{-\frac{1}{2}}\right) dx = \left(\boxed{2}\right) = x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = x + 2 x^{\frac{1}{2}} = \boxed{x + 2\sqrt{x} + C}$$

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e) $f: y = \dfrac{x^3 + 3x^2 - 5}{x^2}$ ... $\dfrac{x^3}{x^2} + \dfrac{3x^2}{x^2} - \dfrac{5}{x^2}$ ... $x + 3 - 5x^{-2}$

$$\int \left(x + 3 - 5x^{-2}\right) dx = \frac{x^2}{2} + 3x - 5 \cdot \frac{x^{-1}}{-1} = \boxed{\frac{1}{2} x^2 + 3x + \frac{5}{x} + C}$$

Příklad 5: Určete křivku, která prochází bodem $A[-2; 1]$ a jejíž sečna v libovolném bodě $T[x; y]$ má směrnici $2x+5$.

Řešení: Směrnice sečny je derivace funkce $f: y = 2x+5$.

$$\int (2x + 5)\, dx = 2 \frac{x^2}{2} + 5x = x^2 + 5x + C$$

$y = x^2 + 5x + C$ ... $A[\underset{x}{-2}; \underset{y}{1}]$ (*)

$1 = (-2)^2 + 5 \cdot (-2) + C$

$1 = 4 - 10 + C$

$\boxed{C = 7}$

dosadíme do (*)

$y = x^2 + 5x + 7$

Křivka je určena rovnicí

$y = x^2 + 5x + 7$.

Příklad 6: Vypočtěte:

a) $\int (x^2 + x + 1)\, dx = \boxed{\dfrac{x^3}{3} + \dfrac{x^2}{2} + x + C}$ ... $\dfrac{1}{3} x^3 + \dfrac{1}{2} x^2 + x + C$

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b) $\int (x^3 - 3x)\, dx = \dfrac{x^4}{4} - 3 \cdot \dfrac{x^2}{2} = \boxed{\dfrac{1}{4} x^4 - \dfrac{3}{2} x^2 + C}$

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c) $\int (1-4x-2x^2)\,dx = x - 4\cdot\frac{x^2}{2} - 2\,\frac{x^3}{3} = \boxed{x - 2x^2 - \frac{2}{3}x^3 + c}$

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d) $\int (2x^4 - 4x^2)\,dx = 2\cdot\frac{x^5}{5} - 4\cdot\frac{x^3}{3} = \boxed{\frac{2}{5}x^5 - \frac{4}{3}x^3 + C}$

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e) $\int x\left(x - \frac{1}{x^2}\right)dx = \int \left(x^2 - \frac{1}{x}\right)dx = \frac{x^3}{3} - \ln|x| + C$ nutno rozvést

podle $\boxed{4}$

$x>0$: $\underbrace{\frac{1}{3}x^3 - \ln x + C}_{x\in(0,+\infty)}$

$x<0$: $\underbrace{\frac{1}{3}x^3 - \ln(-x)}_{x\in(-\infty,0)}$

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f) $\int \frac{x^2+3x+1}{x^2}\,dx = \int\left(\frac{x^2}{x^2} + \frac{3x}{x^2} + \frac{1}{x^2}\right)dx = \int\left(1 + \frac{3}{x} + \frac{1}{x^2}\right)dx =$

$= \int\left(1 + 3\cdot\frac{1}{x} + x^{-2}\right)dx = x + 3\cdot\ln|x| + \frac{x^{-2+1}}{-2+1} = x + 3\ln|x| + \frac{x^{-1}}{-1} =$

$= x + 3\ln|x| - \frac{1}{x} =$

$\rightarrow \boxed{x + 3\ln x - \frac{1}{x} + c \quad \text{pro } x\in(0,+\infty)}$ … $x>0$

$\rightarrow \boxed{x + 3\ln(-x) - \frac{1}{x} + c \quad \text{pro } x\in(-\infty,0)}$ … $x<0$

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g) $\int \frac{1-x^2}{x}\,dx = \int\left(\frac{1}{x} - \frac{x^2}{x}\right)dx = \int\left(\frac{1}{x} - x\right)dx = \ln|x| - \frac{x^2}{2} = \ln|x| - \frac{1}{2}x^2 + c$

$= \rightarrow \boxed{\ln x - \frac{1}{2}x^2 + c}$

$\rightarrow \boxed{\ln(-x) - \frac{1}{2}x^2 + c}$

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h) $\int (1+\sqrt{x})\,dx = \int \left(1 + x^{\frac{1}{2}}\right)dx = x + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = x + \frac{2}{3}x^{\frac{3}{2}} = x + \frac{2}{3}\sqrt{x^3} = \boxed{\frac{2}{3}x\sqrt{x} + C}$

pro $x \in \langle 0,+\infty)$

---

i) $\int \left(\sqrt[3]{x} - 2x\right)dx = \int\left(x^{\frac{1}{3}} - 2x\right)dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} - 2\,\frac{x^2}{2} = \frac{3}{4}\cdot\sqrt[3]{x^4} - x^2 =$

$= \boxed{\frac{3}{4}x\cdot\sqrt[3]{x} - x^2 + C}$

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j) $\int \frac{\sqrt{x} - x}{\sqrt[3]{x}}\,dx = \int\left(\frac{\sqrt{x}}{\sqrt[3]{x}} - \frac{x}{\sqrt[3]{x}}\right)dx = \int\left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} - \frac{x}{x^{\frac{1}{3}}}\right)dx$

$$= \int \left(x^{\frac{1}{2}} \cdot x^{-\frac{1}{3}} - x^{1} \cdot x^{-\frac{1}{3}}\right) dx = \int \left(x^{\frac{1}{6}} - x^{\frac{2}{3}}\right) dx = \frac{x^{\frac{7}{6}}}{\frac{7}{6}} - \frac{x^{\frac{5}{3}}}{\frac{5}{3}} =$$

$$= \frac{6}{7} \cdot x^{\frac{7}{6}} - \frac{2}{5} \cdot x^{\frac{5}{3}} = \frac{6}{7} \cdot \sqrt[6]{x^7} - \frac{3}{5} \cdot \sqrt[3]{x^5} = \boxed{\frac{6}{7} x \sqrt[6]{x} - \frac{3}{5} x \sqrt[3]{x^2} + C}$$

---

k) $$\int \left(\sqrt[4]{x} - \frac{1}{\sqrt[3]{x}}\right) dx = \int x^{\frac{1}{4}} - x^{-\frac{1}{3}} = \frac{x^{\frac{5}{4}}}{\frac{5}{4}} - \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{4}{5} \sqrt[4]{x^5} - \frac{3}{2} \sqrt[3]{x^2} =$$

$$= \boxed{\frac{4}{5} x \sqrt[4]{x} - \frac{3}{2} \sqrt[3]{x^2} + C \quad , \; x \in (0 + \infty)}$$

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l) $\int (2^x + 3^x) dx$ podle $\boxed{7}$ $= \boxed{\frac{2^x}{\ln 2} + \frac{3^x}{\ln 3} + C}$

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m) $$\int (4^x + x^4) dx = \frac{4^x}{\ln 4} + \frac{x^5}{5} = \boxed{\frac{4^x}{\ln 4} + \frac{1}{5} x^5 + C}$$

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n) $\int a \cdot e^x \, dx = \boxed{a e^x + C}$ podle $\boxed{6}$

Příklad 6: Vypočítejte:

a) $\int (5 - x - 3 \sin x) dx = \boxed{5x - \frac{x^2}{2} + 3 \cos x + C}$

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b) $$\int \frac{2a+1}{3 \cos^2 x} dx = \int \left(\frac{2a+1}{3} \cdot \frac{1}{\cos^2 x}\right) dx = \frac{2a+1}{3} \cdot \int \frac{1}{\cos^2 x} dx = \boxed{\frac{2a+1}{3} \cdot \operatorname{tg} x + C},$$

kde $a \in \mathbb{R}$.

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c) $$\int \frac{7k}{5 \cdot \sin^2 x} dx = \int \left(\frac{7k}{5} \cdot \frac{1}{\sin^2 x}\right) dx = \frac{7k}{5} \cdot \int \frac{1}{\sin^2 x} = \frac{7k}{5} \cdot (-\operatorname{cotg} x) + C =$$

$= \boxed{-\frac{7k}{5} \operatorname{cotg} x + C}$, kde $k \in \mathbb{R}$

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d) $$\int \left(2e^x - 2^x - \frac{2}{x}\right) dx = \int \left(2e^x - 2^x - 2 \cdot \frac{1}{x}\right) dx = 2e^x - \frac{2^x}{\ln 2} - 2 \ln|x| + C$$

$$= \begin{cases} \boxed{2e^x - \dfrac{2^x}{\ln 2} - 2 \ln x \quad \text{pro } x > 0} \\ \boxed{2e^x - \dfrac{2^x}{\ln 2} - 2 \ln(-x) \quad \text{pro } x < 0} \end{cases}$$

e) $\int (2\cos x + \sin x - k)\,dx = \boxed{2\sin x - \cos x - kx + c}$, kde $k \in \mathbb{R}$

f) $\int \mathrm{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} - 1\right)dx =$

$= \boxed{\mathrm{tg}\,x - x + c}$

g) $\int \frac{\cos 2x}{\cos^2 x}\,dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x}\,dx = \int \left(\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right)dx = \int \left(1 - \frac{\sin^2 x}{\cos^2 x}\right)dx =$

$\int \left(1 - \frac{1-\cos^2 x}{\cos^2 x}\right)dx = \int \left(1 - \frac{1}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}\right)dx = \int \left(1 - \frac{1}{\cos^2 x} + 1\right)dx =$

$= x - \mathrm{tg}\,x + x + c = \boxed{2x - \mathrm{tg}\,x + c}$

h) $\int \frac{10}{\sin^2 x \cos^2 x}\,dx = 10 \int \frac{1}{\sin^2 x \cos^2 x}\,dx = 10 \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,dx =$

$= 10 \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}\right)dx = 10 \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx =$

$= \boxed{10(\mathrm{tg}\,x - \mathrm{cotg}\,x) + c}$

i) Řešte pomocí vzorce $\left|\cos\frac{x}{2}\right| = \sqrt{\frac{1+\cos x}{2}}$

$\int \cos^2 \frac{x}{2}\,dx = \int \frac{1+\cos x}{2}\,dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos x\right)dx = \boxed{\frac{1}{2}x + \frac{1}{2}\sin x + c}$

Příklad 7: Vypočítejte

a) $\int \frac{x^4 - x^3 + x^2 + x - 2}{x^2 - 1}\,dx =$

$(x^4 - x^3 + x^2 + x - 2) : (x^2 - 1) = x^2 - x + 2$

$\pm x^4 \quad \mp x^2$

$-x^3 + 2x^2 + x - 2$

$\mp x^3 \quad \pm x$

$2x^2 - 2$

$\pm 2x^2 \mp 2$

0

$= \int (x^2 - x + 2)\,dx$

$= \boxed{\frac{x^3}{3} - \frac{x^2}{2} + 2x + c}$

b) $\int \left(2 - \frac{1}{x^3}\right) \cdot \sqrt[3]{x\sqrt{x}}\,dx = \int (2 - x^{-3}) \cdot \sqrt[3]{\sqrt[2]{x^3}}\,dx = \int (2 - x^{-3}) \cdot \sqrt[6]{x^3}\,dx =$

$= \int (2 - x^{-3}) \cdot x^{\frac{1}{2}}\,dx = \int \left(2x^{\frac{1}{2}} - x^{-\frac{5}{2}}\right)dx =$

$$= 2\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} = 2\cdot\frac{2}{3}\cdot x^{\frac{3}{2}} + \frac{2}{3}x^{-\frac{3}{2}} = \frac{4}{3}\sqrt{x^3} + \frac{2}{3}\cdot\frac{1}{\sqrt{x^3}} =$$

$$= \boxed{\frac{4}{3}x\sqrt{x} + \frac{2}{3}\cdot\frac{1}{\sqrt{x^3}} + c}$$

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c) $$\int \frac{x}{x+5}\,dx = \int \frac{x+5-5}{x+5}\,dx = \int\left(\frac{x+5}{x+5} - \frac{5}{x+5}\right)dx = \int\left(1 - \frac{5}{x+5}\right)dx =$$

$$= \int\left(1 - 5\cdot\frac{1}{x+5}\right)dx = x - 5y + c_1 \text{ kde } y = \cancel{x}+5\ldots\ \boxed{x - 5\ln(x+5) + c_1}$$

kde $x > -5$

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d) $$\int \frac{1}{1+\cos 2x}\,dx = \quad \text{vzorec: } \cos 2x = 1 - 2\sin^2 x$$

$$= \int \frac{1}{1+1-2\sin^2 x}\,dx = \int \frac{1}{2-2\sin^2 x}\,dx = \int \frac{1}{2(1-\sin^2 x)}\,dx =$$

$$= \int \frac{1}{2}\cdot\frac{1}{\cos^2 x}\,dx = \boxed{\frac{1}{2}\cdot \operatorname{tg} x + C}$$

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e) $$\int\left(\frac{3}{\sqrt{x}} + \frac{5}{\sqrt[3]{x^2}} + \cos x\right)dx = \int\left(3\cdot x^{-\frac{1}{2}} + 5x^{-\frac{2}{3}} + \cos x\right)dx =$$

$$= 3\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 5\,\frac{x^{\frac{1}{3}}}{\frac{1}{3}} + \sin x = 2\cdot 3\sqrt{x} + 3\cdot 5\sqrt[3]{x} + \sin x$$

$$= \boxed{6\sqrt{x} + 15\sqrt[3]{x} + \sin x + C}$$

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f) $$\int \operatorname{cotg}^2 x\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx = \int\left(\frac{1}{\sin^2 x} - 1\right)dx =$$

$$= \boxed{-\operatorname{cotg} x - x + C}$$

INTEGRAČNÍ METODY: 1) PER PARTES (PO ČÁSTECH) .. pro součin

2) SUBSTITUČNÍ

Vzorce (z nich použijeme ten, který je nejvhodnější).

1.	$\int u'v = uv - \int uv'$	$\rightarrow$ A
2.	$\int vu' = \mathbf{vu} - \int v'u$	

1.	$\int uv' = uv - \int u'v$	$\rightarrow$ B
2.	$\int v'u = \mathbf{vu} - \int vu'$	

Lze psát jako jeden vzorec: $\boxed{\int uv' = uv - \int u'v}$

Příklad 8 (5/143–uč): Určete primitivní funkce k daným funkcím:

a) $\int \underbrace{x}\cdot\underbrace{\sin x}\, dx$

$\mathbf{u}=x$	$\mathbf{v'}=\sin x$
$u'=1$	$v=-\cos x$

$\int uv' = uv - \int u'v$

Tlustě vyznačené proměnné určí typ vzorce, který použijeme. V tomto případě je to vzorec $\boxed{1B}$.

$\int x\cdot\sin x = x\cdot(-\cos x) - \int 1\cdot(-\cos x)\,dx = = -x\cos x - \int(-\cos x)\,dx =$

$= -x\cos + 1\int \cos x\,dx = \boxed{-x\cos x + \sin x + c}$

---

b) $\int \ln x\,dx$ ... $\ln x$ musíme vyjádřit jako součin.

$\int \underbrace{1}\cdot\underbrace{\ln x}\,dx$

$\mathbf{v'}=1$	$\mathbf{u}=\ln x$
$v=x$	$u'=\frac{1}{x}$

$\rightarrow$ Použijeme vzorec $\boxed{2B}$.

$\int v'u = \mathbf{vu} - \int vu'$ ... $\int \ln x\,dx = \int 1\cdot\ln x\,dx = x\cdot\ln x - \int x\cdot\frac{1}{x}\,dx =$

$= x\ln x - \int 1\,dx = \boxed{x\cdot\ln x - x + c}$; $x>0$

---

c) $\int x^2 e^x\,dx$ Metodu per partes použijeme dvakrát.

$\mathbf{u}=x^2,\ \mathbf{v'}=e^x \rightarrow \boxed{1B}$ ... $\int uv' = uv - \int u'v$

$u'=2x,\ v=e^x$ $\int x^2e^x\,dx = x^2e^x - \int 2x\cdot e^x\,dx = x^2e^x - 2\int x\cdot e^x\,dx$ ..

.... pokračování na str. 10.

$\dots x^2e^x - \underbrace{2\int x \cdot e^x dx}_{\substack{u=x \;\; v'=e^x \\ u'=1 \;\; v=e^x}} = \dots$ pomocí vzorce $\boxed{1B}$ ... $\int uv' = uv - \int u'v$

$= x^2e^x - 2\left(x \cdot e^x - \int 1e^x\right) = x^2e^x - 2\left(xe^x - \int e^x\right) =$

$= x^2e^x - 2(xe^x - e^x) = x^2e^x - 2xe^x + 2e^x$

$= \boxed{e^x(x^2 - 2x + 2) + c}$

---

d) $\int x^3 \ln x \, dx$ podle $\int uv = uv - \int uv$

$u' = x^3 \quad v = \ln x \; \dots \boxed{1A}$

$u = \frac{x^4}{4} \quad v' = \frac{1}{x}$

$\int x^3 \ln x = \frac{x^4}{4} \cdot \ln x - \int \left(\frac{x^4}{4} \cdot \frac{1}{x}\right) dx =$

$= \frac{1}{4}x^4 \ln x - \int \frac{x^3}{4} dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx =$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4} \cdot \frac{x^4}{4} = \underbrace{\frac{1}{4}}_{\frac{4}{16}} x^4 \ln x - \frac{x^4}{16} = \boxed{\frac{x^4}{16}(4\ln x - 1) + c}$

Příklad 9 (5.8/144 – uč.): Vypočítejte:

a) $\int x \cos x \, dx$ Pomocí: $\int uv' = uv - \int u'v$

$u = x \quad v' = \cos x$

$u' = 1 \quad v = \sin x$

$\int x \cdot \cos x = x \cdot \sin x - \int (1 \cdot \sin x) dx =$

$= x \cdot \sin x - \int \sin x \, dx = x \cdot \sin x - (-\cos x) =$

$= \boxed{x \sin x + \cos x + C}$

---

b) $\int x \cdot e^x$ .. $\int uv' = uv - \int u'v$

$u = x \quad v' = e^x$

$u' = 1 \quad v = e^x$

$\int x \cdot e^x dx = x \cdot e^x - \int (1 \cdot e^x) dx = x \cdot e^x - \int e^x =$

$= xe^x - e^x = \boxed{e^x(x-1) + C}$

---

c) $\int x^2 \sin x \, dx$ ... $\int uv' = uv - \int u'v$

$u = x^2 \quad v' = \sin x$

$u' = 2x \quad v = -\cos x$

$\int x^2 \sin x \, dx = -x^2 \cos x - \int [2x \cdot (-\cos x)] \, dx =$

$= -x^2 \cos x - \int -2x \cos x \, dx = -x^2 \cos x + 2\int x \cdot \cos x \, dx$

$u = x \quad v' = \cos x$

$u' = 1 \quad v = \sin x$

podle $\int uv' = uv - \int u'v \dots -x^2 \cos x + 2[x \cdot \sin x - \int (1 \cdot \sin x) dx =$

$= -x^2 \cos x + 2[x \cdot \sin x - (-\cos x)] = \boxed{-x^2 \cos x + 2x \sin x + 2\cos x + C}$

d) $\int x e^{2x} dx$ Použijeme vzorce, které v učebnici nejsou:

$$\boxed{u' = e^{ax},\ u = \frac{1}{a}e^{ax} \quad \Big| \quad \int e^{ax} = \frac{1}{a}e^{ax}}$$

$\int x \cdot e^{2x}$ ... $\int vu' = vu - \int v'u$ $\boxed{2A}$ odlišné na označení u, v

$v = x \quad u' = e^{2x}$

$v' = 1 \quad u = \frac{1}{2}e^{2x}$

$$\int x e^{2x} dx = \frac{1}{2}e^{2x} \cdot x - \int 1 \cdot \frac{1}{2}e^{2x} dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x} =$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\cdot\frac{1}{2}e^{2x} = \frac{1}{2}x \cdot e^{2x} - \frac{1}{4}e^{2x} =$$

$$= e^{2x}\left(\frac{x}{2} - \frac{1}{4}\right) = e^{2x} \cdot \frac{2x-1}{4} = \boxed{\frac{1}{4}e^{2x}(2x-1) + c}$$

Příklad 10 (5.91145-uč.): Vypočtěte:

a) $\int x^2 \ln x \, dx$

$v' = x^2 \quad u = \ln x$

$v = \frac{x^3}{3} \quad u' = \frac{1}{x}$

$$\int v'u = vu - \int vu' \quad .. \quad \int x^2 \ln x \, dx =$$

$$= \frac{x^3}{3} \cdot \ln x - \int \left(\frac{x^3}{3} \cdot \frac{1}{x}\right) dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 =$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \frac{x^3}{3} = \frac{x^3 \ln x}{3} - \frac{x^3}{9} =$$

$$= \frac{3x^3 \ln x - x^3}{9} = \frac{x^3(3\ln x - 1)}{9} = \boxed{\frac{x^3}{9}(3\ln x - 1) + c}$$

---

b) $\int x \ln^2 x \, dx$

$v' = x \quad u = \ln^2 x$

$v = \frac{x^2}{2} \quad u'$ derivuj jako složenou funkci

$u' = \frac{2}{x} \cdot \ln x$

$\rightarrow m = \ln x,\ m^2 = 2m \cdot m' = 2\ln x \cdot (\ln x)' =$

$= 2\ln x \cdot \frac{1}{x} = \frac{2}{x} \cdot \ln x$

$\int v'u = vu - \int vu'$

$$\int x \ln^2 x \, dx = \ln^2 x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{2}{x} \ln x \, dx =$$

$$= \ln^2 x \cdot \frac{x^2}{2} - \int x \ln x$$ ... podle $\int v'u = vu - \int vu'$

$v' = x \quad u = \ln x$

$v = \frac{x^2}{2} \quad u' = \frac{1}{x}$

$$= \frac{x^2}{2}\ln^2 x - \left[\frac{x^2}{2}\ln x - \int \left(\frac{x^2}{2} \cdot \frac{1}{x}\right) dx\right] = \frac{x^2}{2}\ln^2 x - \left[\frac{x^2}{2}\ln x - \frac{1}{2}\int x \, dx\right] =$$

$$= \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{1}{2} \cdot \frac{x^2}{2} = \boxed{\frac{x^2}{2}\left(\ln^2 x - \ln x + \frac{1}{2}\right) + c}$$

c) $\int \frac{\ln x}{x^2}dx = \int \ln x \cdot x^{-2} dx = -\frac{1}{x}\ln x - \int \frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx =$ níže

$u = \ln x \quad v' = x^{-2}$

$u' = \frac{1}{x} \quad v = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x} \quad v = -\frac{1}{x}$

$= -\frac{1}{x}\ln x + \int \frac{1}{x^2}dx = -\frac{1}{x}\ln x + \int x^{-2}dx = -\frac{1}{x}\ln x + \frac{x^{-1}}{-1} =$

$= -\frac{1}{x}\ln x - \frac{1}{x} = \boxed{-\frac{1}{x}(\ln x + 1) + C}$

---

d) $\int \ln^2 x = \int 1 \cdot \ln^2 x = x \cdot \ln^2 x - \int x \cdot \frac{2}{x}\ln x\,dx = x\ln^2 x - 2\int \ln x =$ (níže)

$v' = 1 \quad u = \ln^2 x$

$v = x \quad u'$ derivuj jako složenou funkci:

$m = \ln x,\ m^2{}' = 2m \cdot m' = 2\ln x \cdot (\ln x)' =$

$= 2\ln x \cdot \frac{1}{x} = \frac{2}{x}\ln x \quad u' = \frac{2}{x}\ln x$

$= x\ln^2 x - 2\int 1 \cdot \ln x$ ... podle $\int v'u = vu - \int vu'$

$v' = 1 \quad u = \ln x$

$v = x \quad u' = \frac{1}{x}$

$= x\ln^2 x - 2\left[x \cdot \ln x - \int 1\,dx\right] = x\ln^2 x - 2[x \cdot \ln x - x] =$

$= x\ln^2 x - 2x\ln x - 2x = \boxed{x(\ln^2 x - 2\ln x - 2x) + C}$

Příklad 11 (5.10/145-uč.) : Vypočtěte :

a) $\int e^x \sin x\,dx$ pomocí $\int u'v = uv - \int uv'$

$u' = e^x \quad v = \sin x$

$u = e^x \quad v' = \cos x$

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x = \ldots \int u'v = uv - \int uv'$

$u' = e^x \quad v = \cos x$

$u = e^x \quad v' = (-\sin x)$

$= e^x \sin x - \left[e^x \cos x - \int e^x \cdot (-\sin x)dx\right] = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

Dospěli jsme k rovnici

$\underbrace{\int e^x \sin x\,dx}_{\text{neznámá}} = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x\,dx}_{\text{neznámá}}$

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x) \quad /\cdot\tfrac{1}{2}$$

$$\underbrace{\int e^x \sin x\,dx}_{\text{to je zadání}} = \underbrace{\boxed{\tfrac{1}{2} e^x(\sin x - \cos x) + c}}_{\text{to je výsledný integrál}}$$

---

b) $\boxed{\int \cos^2 x\,dx} = \int \cos x \cdot \cos x\,dx =$ podle $\int u'v = uv - \int uv'$

$u' = \cos x \quad v = \cos x$

$u = \sin x \quad v' = -\sin x$

$= \sin x \cos x - \int -\sin^2 x = \sin x \cos x + \int \sin^2 x = \sin x \cos x + \int \sin^2 x =$

$= \sin x \cos x + \int (1-\cos^2 x)\,dx = \sin x \cos x + \int 1\,dx - \int \cos^2 x =$

$= \sin x \cos x + x \boxed{- \int \cos^2 x\,dx}$

Dospěli jsme k rovnici

$$\int \cos^2 x\,dx = \sin x \cos x + x - \int \cos^2 x\,dx$$

$$2\int \cos^2 x\,dx = \sin x \cos x + x \quad /\cdot\tfrac{1}{2}$$

$$\underbrace{\int \cos^2 x\,dx}_{\text{zadání}} = \underbrace{\boxed{\tfrac{1}{2}(\sin x \cos x + x) + c}}_{\text{výsledný integrál}}$$

---

c) $\int \sin(\ln x)\,dx = \int 1 \cdot \sin(\ln x)\,dx$

$u' = 1 \quad v = \sin(\ln x)$ derivuji jako složenou f.

$u = x \quad v' = (\sin m)' = \cos m \cdot m' = \cos(\ln x)\cdot\frac{1}{x} \qquad m = \ln x$

$v' = \frac{1}{x}\cos(\ln x)$

$\boxed{\int \sin(\ln x)\,dx} = x\cdot\sin(\ln x) - \int x \cdot \frac{1}{x}\cos(\ln x)\,dx = x\cdot\sin(\ln x) - \int \cos(\ln x)\,dx =$

$= x\cdot\sin(\ln x) - \int 1 \cdot \cos(\ln x) =$

$u' = 1 \quad v = \cos(\ln x) \quad \ldots\ u = \ln x$

$u = x \quad v' = [\cos(\ln x)]' = -\sin(\ln x)\cdot\frac{1}{x} \quad \ldots\ v' = -\frac{1}{x}\sin(\ln x)$

$=$ ...

$= x\sin(\ln x) - [x\cos(\ln x) - \int x[-\frac{1}{x}\sin(\ln x)\,dx] =$

$= x\sin(\ln x) - x\cos(\ln x) \boxed{- \int \sin(\ln x)\,dx}$

$\ldots$ ⊜

Dospějeme k rovnici: $\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$

$2\int \sin(\ln x)dx = x[\sin(\ln x) - \cos(\ln x)] \quad |:2$

$\underbrace{\int \sin(\ln x)dx}_{\text{zadání}} = \boxed{\frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C}$

Výsledný integrál

---

d) $\boxed{\int e^x \cos x\, dx} = e^x\cos x - \int e^x(-\sin x)dx = e^x\cos x + \int e^x \sin x\, dx =$

$u' = e^x \quad v = \cos x$

$u = e^x \quad v' = -\sin x$

$\int u'v = uv - \int uv'$

$u' = e^x \quad v = \sin x$

$u = e^x \quad v' = \cos x$

$= e^x\cos x + [e^x\sin x - \int e^x\cos x] =$

$= e^x\cos x + e^x\sin x \boxed{- \int e^x\cos x}$

⊜

Dospějeme k rovnici:

$\int e^x\cos x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x$

$2\int e^x\cos x\,dx = e^x\cos x + e^x\sin x$

$2\int e^x\cos x\,dx = e^x(\cos x + \sin x) \quad \cdot\frac{1}{2}$

$\int e^x\cos x\,dx = \boxed{\frac{1}{2}e^x(\sin x + \cos x) + C}$

Zadání — Výsledek

Integrování substituční metodou používáme v těch případech, když je možné integrovanou funkci rozložit na takové dva činitele, z nichž jeden je složená funkce a druhý je její derivace.

Příklad 12 (6/146 - uč.) Vypočítejte $\int (3x-4)^7 dx$

Řešení: $\int (3x-4)^7 dx =$

Označme: sub. $t = 3x-4$

t' označme dt; $\boxed{dt = 3}$

$$= \int (3x-4)^7 \cdot \underbrace{3 \cdot \frac{1}{3}}_{} \, dx$$

aby se nezměnila hodnota dané funkce

$$= \frac{1}{3}\int \underbrace{(3x-4)^7}_{=t} \cdot \underbrace{3 \, dx}_{=dt} = \frac{1}{3}\int t^7 dt = \frac{1}{3} \cdot \frac{t^8}{8} = \frac{1}{24} \cdot t^8 = \boxed{\frac{1}{24}(3x-4)^8 + C}$$

Příklad 13 (5 M/149 - uč.):

a) $\int 10x\,(x^2+13)^{12} dx = 10\int x\,(x^2+13)^{12} dx =$

Sub. $t = x^2+13$

$dt = 2x$

derivace funkce t podle proměnné x

$$= 10\int x\,(x^2+13)^{12} \cdot 2x \cdot \frac{1}{2x}\, dx =$$

$$= 10\int x (x^2+13)^{12} \cdot 2x \cdot \frac{1}{2} \cdot \frac{1}{x}\, dx$$

„vytkneme"

$$= 10 \cdot \frac{1}{2}\int \underbrace{x \cdot \frac{1}{x}}_{=1} (x^2+13)^{12} \cdot 2x\, dx = 5\int \underbrace{(x^2+13)}_{t}{}^{12} \cdot \underbrace{2x}_{=dt} = 5\int t^{12} dt =$$

$$= 5 \cdot \frac{t^{13}}{13} = \frac{5}{13} t^{13} = \boxed{\frac{5}{13}(x^2+13)^{13} + C}$$

b) $\int 2 \sin x \cos^3 x\, dx =$

Sub. $t = \cos x$ $(\cos^3 x = t^3)$

$dt = -\sin x$

$-dt = \sin x$

$$= 2\int \underbrace{\sin x}_{-dt} \, \underbrace{\cos^3 x}_{t^3}\, dx = 2\int -dt \cdot t^3 =$$

$$= -2\int t^3 dt = -2 \cdot \frac{t^4}{4} = -\frac{1}{2} t^4 = \boxed{-\frac{1}{2}\cos^4 x + C}$$

c) $\int 5x^2 e^{x^3} dx =$ Sub. $t = x^3$, $dt = 3x^2$

$= 5\int x^2 e^{x^3} dx = 5\int x^2 \cdot e^{x^3} \cdot 3x^2 \cdot \frac{1}{3x^2} dx = 5\int x^2 \cdot e^{x^3} \cdot 3x^2 \cdot \frac{1}{3} \cdot \frac{1}{x^2} =$

$= \frac{1}{3} \cdot 5 \int \underbrace{x^2 \cdot \frac{1}{x^2}}_{=1} \cdot e^{x^3} \cdot 3x^2 = \frac{5}{3} \int e^{x^3} \cdot \underbrace{3x^2 dx}_{dt} = \frac{5}{3} \int e^t dt = \frac{5}{3} e^t =$

$= \boxed{\frac{5}{3} e^{x^3} + c}$

d) $\int \frac{\ln^2 x}{x} dx = \int \frac{\ln^2 x}{x} \cdot \frac{1}{x} \cdot \frac{x}{1} dx =$ Sub. $t = \ln x$, $dt = \frac{1}{x}$

$= \int \left(\frac{\ln^2 x}{\cancel{x}} \cdot \frac{\cancel{x}}{1} \right) \cdot \frac{1}{x} dx = \int \underbrace{\ln^2 x}_{t^2} \cdot \underbrace{\frac{1}{x} dx}_{dt} = \int t^2 dt = \frac{t^3}{3} = \frac{1}{3} t^3 = \boxed{\frac{1}{3} \ln^3 x + c}$

e) $\int 3x \cdot \sqrt[4]{x^2+5}\, dx = 3\int x \sqrt[4]{x^2+5} \cdot 2x \cdot \frac{1}{2x} dx =$ Sub. $t = x^2+5$, $dt = 2x\, dx$

$= 3\int x \sqrt[4]{x^2+5} \cdot 2x \cdot \frac{1}{2} \cdot \frac{1}{x} dx = 3 \cdot \frac{1}{2} \int \underbrace{x \cdot \frac{1}{x}}_{=1} \sqrt[4]{x^2+5} \cdot 2x = 3 \cdot \frac{1}{2} \int \underbrace{\sqrt[4]{x^2+5}}_{t} \cdot \underbrace{2x\, dx}_{dt} =$

$= \frac{3}{2} \int \sqrt[4]{x^2+5} \cdot 2x\, dx = \frac{3}{2} \cdot \int \sqrt[4]{t}\, dt = \frac{3}{2} \cdot \int t^{\frac{1}{4}} dt = \frac{3}{2} \cdot \frac{t^{\frac{5}{4}}}{\frac{5}{4}} =$

$= \frac{3}{2} \cdot \frac{4}{5} \cdot t^{\frac{5}{4}} = \frac{12}{10} \cdot t^{\frac{5}{4}} = \frac{6}{5} \sqrt[4]{t^5} = \frac{6}{5} \cdot \sqrt[4]{(x^2+5)^5} =$

$= \frac{6}{5} \sqrt[4]{(x^2+5)^4 \cdot (x^2+5)} = \boxed{\frac{6}{5} (x^2+5) \cdot \sqrt[4]{x+5} + c}$

f) $\int \sin^2 x \cos^3 x\, dx =$ Sub. $t = \sin x$, $dt = \cos x$

$= \int \sin^2 x \cdot \cos^2 x \cdot \cos x\, dx = \int \underbrace{\sin^2 x}_{t^2} (1 - \underbrace{\sin^2 x}_{t^2}) \cdot \underbrace{\cos x}_{dt} =$

$= \int t^2 (1 - t^2) dt = \int (t^2 - t^4) dt =$

$= \frac{t^3}{3} - \frac{t^5}{5} = \frac{1}{3} t^3 - \frac{1}{5} t^5 = \boxed{\frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x}$

Příklad 14 (5.13/149):

Sub. $t = 7x$ $dt = 7$

a) $\int \sin 7x\, dx = \int \sin 7x \cdot 7 \cdot \frac{1}{7} dx = \frac{1}{7} \int \sin 7x \cdot 7\, dx = \frac{1}{7} \int \sin t\, dt =$

$= \frac{1}{7} (-\cos t) = \boxed{-\frac{1}{7} \cos 7x + C}$

b) $\int 5k \cos \frac{8}{3}x\, dx = 5k\int \cos\frac{8}{3}x\, dx =$ Sub. $t = \frac{8}{3}x,\ dt = \frac{8}{3}$

$= 5k\int \cos\frac{8}{3}x \cdot \frac{8}{3}\cdot\frac{3}{8}\, dx = 5k\cdot\frac{3}{8}\int \cos\frac{8}{3}x\cdot\frac{8}{3}\, dx = \frac{15k}{8}\cos t\, dt =$

$= \frac{15k}{8}\cdot \sin t = \boxed{\frac{15k}{8}\sin\frac{8}{3}x + c}$

---

c) $\int 3e^{-x}dx = 3\int e^{-x}dx = 3\int e^{x}\cdot(-1)\cdot\left(\frac{1}{-1}\right)dx =$ | Sub. $t=-x,\ dt=-1$

$= -\frac{3}{1}\int e^{-x}\cdot(-1)\,dx = -3\int e^{t}dt = -3e^{t} = \boxed{-3e^{-x}+c}$

---

d) $\int 2e^{3x-1}dx = 2\int e^{3x-1}dx =$ Sub. $t=3x-1,\ dt=3$

$= 2\int e^{3x-1}\cdot 3\cdot\frac{1}{3}\,dx = \frac{2}{3}\int e^{3x-1}\cdot 3\,dx = \frac{2}{3}\int e^{t}dt = \frac{2}{3}e^{t} =$

$= \boxed{\frac{2}{3}e^{3x-1}+c}$

Příklad 15: Sub. $t = 1+3x,\ dt = 3$

a) $\int \sqrt{1+3x}\,dx = \int\sqrt{1+3x}\cdot 3\cdot\frac{1}{3}\,dx = \frac{1}{3}\int (1+3x)^{\frac{1}{2}}\cdot 3\,dx = \frac{1}{3}\int t^{\frac{1}{2}}dt =$

$= \frac{1}{3}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{3}\cdot\frac{2}{3}\cdot t^{\frac{3}{2}} = \frac{2}{9}\sqrt{t^3} = \frac{2}{9}\sqrt{(1+3x)^3} = \frac{2}{9}\sqrt{(1+3x)^2\cdot(1+3x)} =$

$= \boxed{\frac{2}{9}(1+3x)\sqrt{1+3x}+c}$

---

b) $\int 2x\sin(x^2+1)\,dx = 2\int x\cdot\sin(x^2+1)\,dx =$ Sub. $t=x^2+1$, $dt=2x$

$= 2\int x\cdot\sin(x^2+1)\cdot 2x\cdot\frac{1}{2x}\,dx = 2\int\frac{x}{2x}\sin(x^2+1)\cdot 2x\,dx =$

$2\cdot\frac{1}{2}\int\sin(x^2+1)\cdot 2x\,dx = \int\sin t\,dt = -\cos t = \boxed{-\cos(x^2+1)+c}$

---

c) $\int\frac{dx}{\sqrt{x-3}} = \int\frac{1}{\sqrt{x-3}}\,dx = \int(x-3)^{-\frac{1}{2}}dx =$ Sub. $t=x-3$, $dt=1$

$= \int(x-3)^{-\frac{1}{2}}\cdot 1\cdot\frac{1}{1} = \int(x-3)^{-\frac{1}{2}}\cdot 1 = \int t^{\frac{1}{2}}dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = 2\cdot t^{\frac{1}{2}} = 2\sqrt{t} =$

$= \boxed{2\sqrt{x-3}+c}$

---

d) $\int \frac{2x}{5-x^2}\,dx = \int \frac{-dt}{t} =$ Sub. $t = 5-x^2$, $dt = -2x$, $-dt = 2x$

$= -1\int \frac{dt}{t} = -1\cdot\int \frac{1}{t}\,dt = -1\cdot \ln|t| = \boxed{-\ln|5-x^2|+C}$

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e) $\int \frac{-x^2}{2\sqrt[3]{4x^3-11}}\,dx =$

$t = 4x^3-11$, $dt = \underbrace{12x}$

vzhľadom k 12x musíme dostať 12x do čitateľa integrovaného výrazu

$= \int \frac{-x^2}{2\sqrt[3]{4x^3-11}} \cdot \frac{-12}{-12}\,dx =$

$\int \frac{12x}{-24\sqrt[3]{4x^3-11}}\,dx = -\frac{1}{24}\int \frac{12x}{\sqrt[3]{4x^3-11}}\,dx = -\frac{1}{24}\int \frac{dt}{\sqrt[3]{t}} = -\frac{1}{24}\int \frac{1}{t^{\frac{1}{3}}}\,dt =$

$= -\frac{1}{24}\int t^{-\frac{1}{3}}\,dt = -\frac{1}{24}\cdot \frac{t^{\frac{2}{3}}}{\frac{2}{3}} = -\frac{1}{24}\cdot\frac{3}{2}\,t^{\frac{2}{3}} = -\frac{1}{16}\sqrt[3]{t^2} =$

$= \boxed{-\frac{1}{16}\sqrt[3]{(4x^3-11)^2} + C}$

VZOR k riešení úloh typu $\int \cos^4 x \sin^5 x\,dx =$

f) $= \int \cos^4 x \cdot \underline{\sin^2 x} \cdot \underline{\sin^2 x} \cdot \sin x\,dx = \int \cos^4 x \cdot \sin x \cdot (1-\cos^2 x)\cdot(1-\cos^2 x)\,dx =$

$= \int \cos^4 x \cdot (1-\cos^2 x)^2 \cdot \sin x\,dx = \ldots$

Sub. $t = \cos x$
$dt = -\sin x$
$\sin x = -dt$

$\int t^4 \cdot (1-t^2)^2 \cdot (-dt) = -\int t^4 \cdot (1-2t^2+t^4)\,dt =$

$= -\int (t^4 - 2t^6 + t^8)\,dt = -\left(\frac{t^5}{5} - 2\frac{t^7}{7} + \frac{t^9}{9}\right) =$

$= -\frac{1}{5}t^5 + \frac{2}{7}t^7 - \frac{1}{9}t^9 = \boxed{-\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + c}$

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g) $\int \sin^7 x\,dx = \int \sin^2 x \cdot \sin^2 x \cdot \sin^2 x \cdot \sin x\,dx = f$

$= \int (1-\cos^2 x)\cdot(1-\cos^2 x)\cdot(1-\cos^2 x)\cdot \sin x\,dx = \int (1-\cos^2 x)^3 \cdot \sin x\,dx =$

Sub: $t = \cos x$
$dt = -\sin x$
$-dt = \sin x$

$= \int (1-t^2)^3 \cdot (-dt) = -\int (1-t^2)^3\,dt =$

$= -\int (1-3t^2+3t^4-t^6)\,dt =$

$$= -\left(t - 3\cdot\frac{t^3}{3} + \frac{3\cdot t^5}{5} - \frac{t^7}{7}\right) = -t + t^3 - \frac{3}{5}t^5 + \frac{1}{7}t^7 =$$

$$= \boxed{-\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C}$$

h) $\int \frac{30x^2}{10x^3-9}\,dx =$ Sub. $t = 10x^3 - 9$, $dt = 30x^2$

$$= \int \frac{dt}{t} = \int \frac{1}{t}\,dt = \ln|t| = \boxed{\ln|10x^3-9| + C} \quad ; 10x^3-9 \neq 0$$

i) $\int \frac{-7}{2x-7}\,dx = -7\cdot\int \frac{1}{2x-7}\,dx =$ Sub. $t = 2x-7$

$dt = 2$

$$= -7\cdot\int \frac{1}{2x-7}\cdot 2\cdot\frac{1}{2}\,dx = -\frac{7}{2}\int \frac{1}{2x-7}\cdot 2\,dx = -\frac{7}{2}\int \frac{1}{t}\,dt =$$

$$= \boxed{-\frac{7}{2}\ln|t| = -\frac{7}{2}\ln|2x-7| + C}$$

Další příklady:

Příklad 16:

a) $\int (1-x)^2\,dx = \int (1-2x+x^2)\,dx = x - 2\frac{x^2}{2} + \frac{x^3}{3} = \boxed{x - x^2 + \frac{1}{3}x^3 + c}$

b) $\int \left(3\sin x - \frac{5}{\sin^2 x} + 2\cos x - 1\right) dx = 3\cdot(-\cos x) - 5\cdot(-\operatorname{cotg} x + 2\sin x - x =$

$$= \boxed{-3\cos x + 5\operatorname{cotg} x + 2\sin x + x}$$