16 a) $\boxed{\text{MOIVREOVA VĚTA}}$

Věta: Pro každé přirozené číslo $n$ a každé komplexní číslo $r(\cos\varphi + i\sin\varphi)$ platí:

$$[r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos n\varphi + i\sin n\varphi)$$

Moivreova věta: Pro každé přirozené číslo $n$ a libovolné reálné číslo $\varphi$ platí: $\boxed{2}$

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$

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Moivreova věta je zvláštní případ věty

$[|z|(\cos\varphi + i\sin\varphi)]^n = |z|^n(\cos n\varphi + i\sin n\varphi)$, kde $n \in N$ a $|z| = 1$. $\boxed{1}$

---

Příklad 1 (podle u 155-uč.)

a) $(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)^{62} = \cos 62\cdot\frac{1}{3}\pi + i\sin 62\cdot\frac{1}{3}\pi = \cos\frac{62}{3}\pi + i\sin\frac{62}{3}\pi =$

$= \cos(\cancel{20\pi} + \frac{2}{3}\pi) + i\sin(\cancel{20}\pi + \frac{2}{3}\pi) = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi = \boxed{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

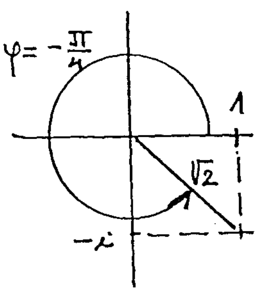
Poznámka: 1) Je-li kompl. číslo zadáno v goniometrickém tvaru, musí výsledek být vyjádřen v algebraickém tvaru.

2) Moivreovu větu jsme užili při umocnění kompl. čísla v gon. tvaru na přirozený exponent.

3) Chceme-li umocnit kompl. číslo v alg. tvaru, obvykle ho nejprve uvedeme na gon. tvar, protože vypočítat mocniny $(a+bi)^n$ pomocí binomické věty je pro větší $n$ zdlouhavé.

---

b) $(1-i)^{100} = \sqrt{2}\left[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right] = \left(\sqrt{2}\left[\cos(-\frac{1}{4}\pi) + i\sin(-\frac{1}{4}\pi)\right]\right)^{100} =$




$= \text{podle } \boxed{1} \ldots (\sqrt{2})^{100}\cdot\left[\cos 100\cdot(-\frac{1}{4}\pi) + i\sin 100\cdot(-\frac{1}{4}\pi)\right] =$

$= (2^{\frac{1}{2}})^{100}\cdot\left[\cos(-25\pi) + i\sin(-25\pi)\right] =$

$= 2^{50}\cdot\left[\cos(-25\pi) + i\sin(-25\pi)\right] = \ldots$ $\quad -25\pi = -\cancel{24\pi} - \pi,$ $\quad -\pi = +\pi$

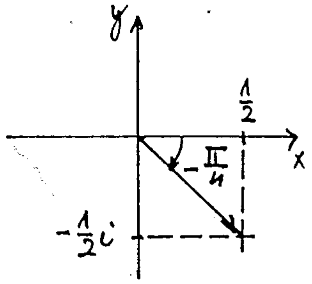
$= 2^{50}(\underbrace{\cos\pi}_{-1} \underset{+}{+} i\underbrace{\sin\pi}_{0}) = \boxed{-2^{50}}$

---

c) $\left(\frac{1}{1+i}\right)^{10}$ řešíme částečně odlišně od učebnice

Výraz $\frac{1}{1+i}$ upravíme: $\frac{1}{1+i} = \underbrace{\frac{1\cdot(1-i)}{(1+i)\cdot(1-i)}}_{\text{VZOREC: } (a+bi)\cdot(a-bi)=a^2+b^2} = \frac{1-i}{1^2+1^2} = \frac{1-i}{2} =$

$= \boxed{\frac{1}{2} - \frac{1}{2}i}$




$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\left(\frac{1}{1+i}\right)^{10} = \left[\frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)\right]^{10} =$ ...VZORCE: $\cos(-x)=\cos x$, $\sin(-x)=-\sin x$

$=\left(\frac{1}{\sqrt{2}}\right)^{10}\cdot\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{10} =$ $\qquad \left(\frac{1}{2^{\frac{1}{2}}}\right)^{10} = \frac{1}{2^5} = \frac{1}{32}$

$= \frac{1}{32}\cdot\left(\cos\frac{10}{4}\pi - \sin\frac{10}{4}\pi\right) =$

$= \frac{1}{32}\cdot\left(\cos\frac{5}{2}\pi - \sin\frac{5}{2}\pi\right) = \frac{1}{32}\left(\cos 2\tfrac{1}{2}\pi - \sin 2\tfrac{1}{2}\pi\right) =$

$= \frac{1}{32}\cdot\left(\underbrace{\cos\frac{\pi}{2}}_{0} - \underbrace{\sin\frac{\pi}{2}}_{-i}\right) = \frac{1}{32}\cdot(-i) = \boxed{-\frac{1}{32}i}$

---

Příklad 2: Vypočítejte:

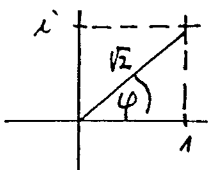
a) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{59} = \cos\frac{59}{6}\pi + i\sin\frac{59}{6}\pi = \cos 9\tfrac{5}{6}\pi + i\sin 9\tfrac{5}{6}\pi =$

$= \cos\left(8\pi + 1\tfrac{5}{6}\pi\right) + i\sin\left(8\pi+1\tfrac{5}{6}\pi\right) = \cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi =$

$= \boxed{\frac{\sqrt{3}}{2} - \frac{1}{2}i}$ Při řešení použít vzorec $\boxed{2}$

---

b) $(1+i)^{10} = \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{10} = (\sqrt{2})^{10}\cdot\left[\cos 10\cdot\frac{\pi}{4} + i\sin 10\cdot\frac{\pi}{4}\right] =$




$\varphi = \frac{\pi}{4}$

$=\left(2^{\frac{1}{2}}\right)^{10}\cdot\left[\cos\frac{10}{4}\pi + i\sin\frac{10}{4}\pi\right] = 2^5\cdot\left(\cos\frac{5}{2}\pi + i\sin\frac{5}{2}\pi\right) =$

$= 32\cdot\left(\cos 2\tfrac{1}{2}\pi + i\sin 2\tfrac{1}{2}\pi\right) = 32\cdot\left(\underbrace{\cos\tfrac{1}{2}\pi}_{0} + i\,\underbrace{\sin\tfrac{1}{2}\pi}_{1}\right) =$

$= 32\cdot i = \boxed{32i}$

---

c) $\left(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi\right)^{31} = \cos\frac{31}{6}\pi + i\sin\frac{31}{6}\pi = \cos 5\tfrac{1}{6}\pi + i\sin 5\tfrac{1}{6}\pi =$

$= \cos\left(4\pi + 1\tfrac{1}{6}\pi\right) + i\sin\left(4\pi+1\tfrac{1}{6}\pi\right) = \cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi = \boxed{-\frac{\sqrt{3}}{2} - \frac{1}{2}i}$

---

d) $(\underbrace{-\sqrt{2}-i\sqrt{2}}_{z})^4 =$

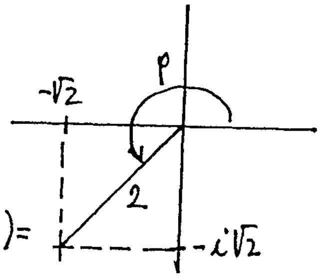
$|z|=\sqrt{(\sqrt{2})^2+(\sqrt{2})^2}=\sqrt{2+2}=2$

$=\left[2\cdot\left(\cos\frac{5}{4}\pi+i\sin\frac{5}{4}\pi\right)\right]^4=$

$=2^4\left(\cos 4\cdot\frac{5}{4}\pi+i\sin 4\cdot\frac{5}{4}\pi\right)=$

$=16\left(\cos 5\pi+i\sin 5\pi\right)=16\left[\cos(4\pi+\pi)+i\sin(4\pi+\pi)\right]=$

$=16\left(\underbrace{\cos\pi}_{-1}+i\underbrace{\sin\pi}_{0}\right)=\boxed{-16}$




$\varphi=\pi+\frac{\pi}{4}=\frac{5}{4}\pi$

e) $(1-i)^5=\left[\sqrt{2}\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)\right]^5=$

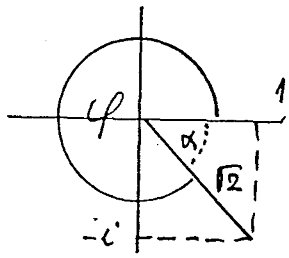
$=\left(2^{\frac{1}{2}}\right)^5\cdot\left[\cos\left(5\cdot\frac{7}{4}\pi\right)+i\sin\left(5\cdot\frac{7}{4}\pi\right)\right]=$

$=2^{\frac{5}{2}}\cdot\left(\cos\frac{35}{4}\pi+i\sin\frac{35}{4}\pi\right)=$

$=\sqrt{2^5}\cdot\left(\cos\cancel{8}\frac{3}{4}\pi+i\sin\cancel{8}\frac{3}{4}\pi\right)=$

$=4\sqrt{2}\cdot\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)=4\sqrt{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)=$

$=-\frac{4\sqrt{2}\cdot\sqrt{2}}{2}+\frac{4\sqrt{2}\cdot\sqrt{2}}{2}i=\boxed{-4+4i}$




$\varphi=2\pi-\frac{\pi}{4}=\frac{7}{4}\pi$

Příklad 3: Vypočítejte $z^9$, je-li $z=\sqrt{3}-i$

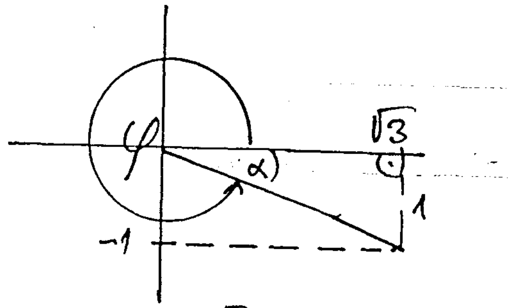
$z=2\left(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi\right)$

$z^9=\left[2\left(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi\right)\right]^9$

$z^9=2^9\cdot\left(\cos\cancel{16}\frac{1}{2}\pi+i\sin\cancel{16}\frac{1}{2}\pi\right)=$

$=512\cdot\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=512\cdot(0+i\cdot 1)=$

$=\boxed{512i}$




$\operatorname{tg}\alpha=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}\Rightarrow\alpha=\frac{\pi}{6}$

$\varphi=2\pi-\frac{\pi}{6}=\frac{11}{6}\pi$

$|z|=\sqrt{(\sqrt{3})^2+1^2}=\sqrt{3+1}=2$

Příklad 4: Vypočítejte $a^5$, je-li

$a=\frac{15-5i}{1+2i}-\frac{1-3i}{i}+(3+i)\cdot(-1+2i)=$

$a=3,\ b=1 \quad c=-1 \quad d=2$

Při řešení použijeme přímo jiný vzorec

$$(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$$

⊗ DOLE

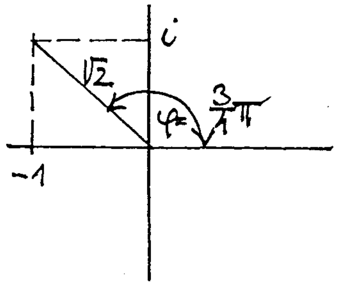
$$= \frac{\overset{a}{(15}-\overset{b}{5i)}\cdot\overset{c}{(1}-\overset{d}{2i)}}{(1+2i)\cdot(1-2i)} - \frac{(1-3i)\cdot(-i)}{i\cdot(-i)} + \left[(-3-2)+(6-1)i\right] =$$

$$= \frac{(15-10)+(-30+5)i}{1+4} - \frac{-3-i}{-i^2} + (-5+5i) =$$

$$= \frac{5-25i}{5} - \frac{-3-i}{1} + (-5+5i) = (1-5i)+(3+i)+(-5+5i) = \boxed{-1+i}$$

Platí tedy, že $a=-1+i$

$$a=\sqrt{2}\left(\cos\tfrac{3}{4}\pi + i\sin\tfrac{3}{4}\pi\right)$$




$$a^5 = \left[\sqrt{2}\left(\cos\tfrac{3}{4}\pi + i\sin\tfrac{3}{4}\pi\right)\right]^5 =$$

$$= (\sqrt{2})^5\left[\cos\left(5\cdot\tfrac{3}{4}\pi\right) + i\sin\left(5\cdot\tfrac{3}{4}\pi\right)\right] =$$

$$= 2^{\frac{5}{2}}\left(\cos\tfrac{15}{4}\pi + i\sin\tfrac{15}{4}\pi\right) = \sqrt{2^5}\cdot\left[\cos\left(\tfrac{8}{4}\pi+\tfrac{7}{4}\pi\right) + i\sin\left(\tfrac{8}{4}\pi+\tfrac{7}{4}\pi\right)\right] =$$

$$2\cdot\sqrt{2}\left[\cos\left(\cancel{2\pi}+\tfrac{7}{4}\pi\right) + i\sin\left(\cancel{2\pi}+\tfrac{7}{4}\pi\right)\right] = 2\sqrt{2}\left(\cos\tfrac{7}{4}\pi + i\sin\tfrac{7}{4}\pi\right) =$$

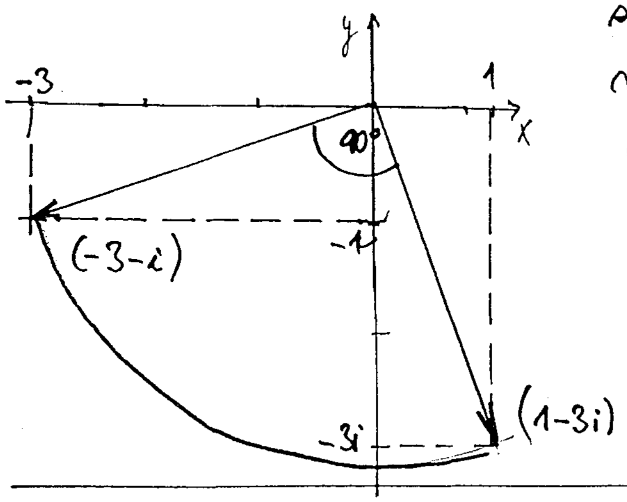
$$= 2\sqrt{2}\left[\tfrac{\sqrt{2}}{2} + \left(-\tfrac{\sqrt{2}}{2}\right)i\right] = 2\sqrt{2}\left(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right) = \frac{2\sqrt{2}\cdot\sqrt{2}}{2} - \frac{2\sqrt{2}\sqrt{2}}{2}i = 2-2i$$

Výsledek: $\boxed{a^5 = 2-2i \quad \text{nebo } a^5 = 2\sqrt{2}\left(\cos\tfrac{7}{4}\pi - \sin\tfrac{7}{4}\pi\right)}$ ukázat na obr.

⊗ Poznámka: násobíme-li kompl. číslo číslem $(-i)$, otočí se jeho obraz o $\left(-\frac{\pi}{2}\right)$, o $(-90°)$.

V našem případě platí:

$$(1-3i)\cdot(-i) = -i+3i^2 = -3-i$$



Příklad 5: Vyjádřete $\sin 3x$ a $\cos 3x$ pomocí součinu $\sin x$ a $\cos x$.

(A) Pomocí Moivreovy věty platí: $(\cos x + i\sin x)^3 = \underline{\cos 3x} + \underline{\underline{i\sin 3x}}$

(B) Pomocí vzorce $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ platí:

$$(\cos x + i\sin x)^3 = \cos^3 x + 3\cos^2 x \sin x + 3\cos x (i\sin x)^2 + (i\sin x)^3 =$$

$$= \cos^3 x + 3\cos^2 x \sin x + 3\cos x(-1\sin^2 x) + i^3 \sin^3 x = \qquad i^3 = -i$$

$$= \underline{\cos^3 x} + \underline{\underline{3\cos^2 x\, i\sin x}} - \underline{3\cos x \sin^2 x} - \underline{\underline{i\sin^3 x}} =$$

$$= \underline{\cos^3 x - 3\cos x \sin^2 x} + \underline{\underline{i(3\cos^2 x \sin x - \sin^3 x)}}$$

Reálné složky k. čísla a imaginární složky k. čísla se rovnají. Proto platí:

$$\boxed{\cos 3x = \cos^3 x - 3\cos x \sin^2 x} \qquad \boxed{\sin 3x = 3\cos^2 x \sin x - \sin^3 x}$$

Příklad 6: Vypočítejte $(\underbrace{1 + \cos \tfrac{1}{4}\pi + i\sin\tfrac{1}{4}\pi}_{x})^{12}$

Při řešení užijeme vzorce

$$\boxed{1 + \cos\alpha = 2\cos^2 \tfrac{1}{2}\alpha} \ \text{(I.)} \qquad \boxed{\sin\alpha = 2\sin\tfrac{1}{2}\alpha \cos\tfrac{1}{2}\alpha} \ \text{(II.)}$$

$$x = (1 + \cos\tfrac{1}{4}\pi + i\sin\tfrac{1}{4}\pi) = 2\cos^2\tfrac{1}{8}\pi + i2\sin\tfrac{1}{8}\pi\cos\tfrac{1}{8}\pi =$$

Vytkneme $2\cos\frac{1}{8}\pi$

$$= 2\cos\tfrac{1}{8}\pi(\cos\tfrac{1}{8}\pi + i\sin\tfrac{1}{8}\pi)$$

$$x^{12} \ldots (1 + \cos\tfrac{1}{4}\pi + i\sin\tfrac{1}{4}\pi)^{12} = [2\cos\tfrac{1}{8}\pi(\cos\tfrac{1}{8}\pi + i\sin\tfrac{1}{8}\pi)]^{12} =$$

$$= 2^{12}\cos^{12}\tfrac{1}{8}\pi(\cos\tfrac{12}{8}\pi + i\sin\tfrac{12}{8}\pi) = \boxed{2^{12}\cos^{12}\tfrac{1}{8}\pi(\cos\tfrac{3}{2}\pi + i\sin\tfrac{3}{2}\pi)}$$

Příklad 7: Vypočítejte $(\underbrace{1 + \cos\tfrac{1}{3}\pi}_{\text{vzorec (I.)}} + i\underbrace{\sin\tfrac{1}{3}\pi}_{\text{vzorec (II.)}})^{12} = (2\cos\frac{\pi}{6} + 2\sin\frac{\pi}{6} +$

$$+ 2\sin\tfrac{\pi}{6}\cos\tfrac{\pi}{6}i)^{12} = [\underbrace{2\cos\tfrac{\pi}{6}}_{2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}}(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6})]^{12} = [\sqrt{3}(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6})]^{12} =$$

$$(\sqrt{3})^{12}\cdot(\cos 12\cdot\tfrac{\pi}{6} + i\sin 12\cdot\tfrac{\pi}{6}) = (3^{\frac{1}{2}})^{12}\cdot(\cos 2\pi + i\sin 2\pi) =$$

$$= 729(\cos 2\pi + i\sin 2\pi) = 729(1+0i) = \boxed{729}$$

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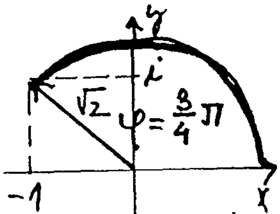
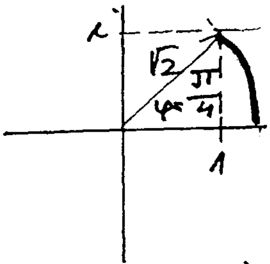
Příklad 8: Vypočítejte:




$$(-1+i)^{66} - i(1+i)^{80} =$$

$$= \left[\sqrt{2}\left(\cos\tfrac{3}{4}\pi + i\sin\tfrac{3}{4}\pi\right)\right]^{66} - i\left[\sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)\right]^{80} =$$

$$= \left(2^{\frac12}\right)^{66}\cdot\left(\cos\tfrac{99}{2}\pi + i\sin\tfrac{99}{2}\pi\right) - i\left(2^{\frac12}\right)^{80}\cdot\left(\cos 20\pi + i\sin 20\pi\right) =$$

$$2^{33}\cdot\left(\cos 49\tfrac12\pi + i\sin 49\tfrac12\pi\right) - i\,2^{40}\cdot\left(\cos 0\pi + i\sin 0\pi\right) =$$

$$2^{33}\cdot\left(\cos\tfrac32\pi + i\sin\tfrac32\pi\right) - i\cdot 2^{40}(1+0) = 2^{33}[0+i(-1)] - 2^{40} =$$

$$= 2^{33}\cdot(-i) - 2^{40}\cdot i = -2^{33}i - 2^{40}i = -2^{33}i\underbrace{(1+2^{7})}_{129} = \boxed{-129\cdot 2^{33}i}$$

---

Příklad 9: Vypočítejte:

$$\left(\frac{i}{1-\sqrt{3}i}\right)^{100} = \left(\frac{i(1+\sqrt{3}i)}{(1-\sqrt{3}i)\cdot(1+\sqrt{3}i)}\right)^{100} = \left(\frac{i+\sqrt{3}i^2}{1^2-(\sqrt{3}i)^2}\right)^{100} = \left(\frac{i-\sqrt{3}}{1+3}\right)^{100} =$$

$$= \left(\frac{-\sqrt{3}+i}{4}\right)^{100} = \left(-\frac{\sqrt{3}}{4}+\frac{i}{4}\right)^{100} = \left[\frac12\left(-\frac{\sqrt{3}}{2}\right)+\frac12\cdot\frac12 i\right]^{100} = \left[\frac12\left(-\frac{\sqrt{3}}{2}+\frac12 i\right)\right]^{100} =$$

Bylo nutné vytknout jen $\frac12$, aby zůstaly hodnoty gon. funkcí, které známe.

$$= \left[\frac12\left(\cos\tfrac56\pi + i\sin\tfrac56\pi\right)\right]^{100} = \left(\frac12\right)^{100}\cdot\left(\cos\tfrac56\pi + i\sin\tfrac56\pi\right)^{100} =$$

$$= 2^{-100}\cdot\left(\cos\tfrac{250}{3}\pi + i\sin\tfrac{250}{3}\pi\right) = 2^{-100}\cdot\left(\cos 83\tfrac13\pi + i\sin 83\tfrac13\pi\right) =$$

$$= 2^{-100}\cdot\left[\cos\left(\cancel{82\pi} + \tfrac43\pi\right) + i\sin\left(\cancel{82\pi} + \tfrac43\pi\right)\right] = \boxed{2^{-100}\cdot\left(\underbrace{\cos\tfrac43\pi}_{-\frac12} + i\underbrace{\sin\tfrac43\pi}_{-\frac{\sqrt{3}}{2}}\right)}$$

nebo $2^{-100}\cdot\left[-\frac12 + i\left(-\frac{\sqrt{3}}{2}\right)\right]$

$$= \boxed{2^{-100}\cdot\left(-\frac12 - \frac{\sqrt{3}}{2}i\right)}$$

Příklad 10: Podle postupu v příkladu 5 vypočítejte $\sin 4x$ a $\cos 4x$.

Podle binomické věty pro exponent 4 platí:

$$(a\pm b)^4 = a^4 + 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$$

$$(\cos x + i \sin x)^4 = \cos^4 x + 4i \cos^3 x \sin x + 6 \cdot \cos^2 x \cdot i^2 \sin^2 x +$$
$$+ 4 \cos x\, i^3 \sin^3 x + i^4 \sin^4 x =$$
$$= \underline{\cos^4 x} + \underline{\underline{4i \cos^3 x \sin x}} - \underline{6 \cos^2 x \sin^2 x} - \underline{\underline{4i \cos x \sin^3 x}} + \underline{\sin^4 x}$$

Podle Moivreovy věty platí:

$$(\cos x + i \sin x)^4 = \underline{\cos 4x} + i\, \underline{\underline{\sin 4x}}$$

$$\boxed{\begin{aligned} \cos 4x &= \cos^4 x - 6 \cos^2 x \sin^2 x + \sin^4 x \\ \sin 4x &= 4 \cos^3 x \sin x - 4 \cos x \sin^3 x \end{aligned}}$$

---

Zopakuj si: $i = \sqrt{-1}$ $\quad i^4 = 1$ $\quad i^7 = -i$ $\quad i^{4k+1} = i$

$i^2 = -1$ $\quad i^5 = i$ $\quad i^8 = 1$ $\quad i^{4k+2} = -1$

$i^3 = -i$ $\quad i^6 = -1$ $\quad i^{4k+3} = -i$

$i^{4k} = 1$

---

Příklad 11: Vypočítejte

a) $\left(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi\right)^{50} = \cos \frac{50}{4}\pi + i \sin \frac{50}{4}\pi =$

$= \cos \cancel{12}\frac{1}{2}\pi + i \sin \cancel{12}\frac{1}{2}\pi = \underbrace{\cos \frac{1}{2}\pi}_{0} + i \underbrace{\sin \frac{1}{2}\pi}_{1} = 0 + i \cdot 1 = \boxed{i}$

---

b) $\left(\cos \frac{1}{6}\pi + i \sin \frac{1}{6}\pi\right)^{31} = \cos \frac{31}{6}\pi + i \sin \frac{31}{6}\pi = \cos 5\frac{1}{6}\pi + i \sin 5\frac{1}{6}\pi =$

$= \cos\left(4\pi + 1\frac{1}{6}\pi\right) + i \sin\left(4\pi + 1\frac{1}{6}\pi\right) = \cos \frac{7}{6}\pi + i \sin \frac{7}{6} = \boxed{-\frac{\sqrt{3}}{2} - \frac{1}{2}i}$

---

c) $\left(2\left[\cos\left(-\frac{1}{4}\pi\right) + i \sin\left(-\frac{1}{4}\pi\right)\right]\right)^{50} = 2^{50} \cdot \left[\cos\left(-\frac{1}{4}\pi\right) + i \sin\left(-\frac{1}{4}\pi\right)\right]^{50} =$

$= 2^{50}\left[\cos\left(-\frac{50}{4}\pi\right) + i \sin\left(-\frac{50}{4}\pi\right)\right] = 2^{50}\left[\cos\left(-\frac{25}{2}\pi\right) + i \sin\left(-\frac{25}{2}\pi\right)\right] =$

$= 2^{50}\left[\cos\left(-12\frac{1}{2}\pi\right) + i \sin\left(-12\frac{1}{2}\pi\right)\right] = \ldots$ $\quad -12\frac{1}{2}\pi = \cancel{-12\pi} - \frac{1}{2}\pi = -\frac{1}{2}\pi$

$2\pi - \frac{1}{2}\pi = \frac{3}{2}\pi$

$$= 2^{50}\left(\cos\tfrac{3}{2}\pi + i\sin\tfrac{3}{2}\pi\right) = 2^{50}[0+i(-1)] = 2^{50}\cdot(-i) = \boxed{-2^{50}i}$$

d) $$\left(\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\right)^{70} = \cos\tfrac{140}{3}\pi + i\sin\tfrac{140}{3}\pi = \cos\cancel{46}\tfrac{2}{3}\pi + i\sin\cancel{46}\tfrac{2}{3}\pi =$$

$$= \cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi = \boxed{-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i}$$

Příklad 12: Vypočítejte $z^8$, je-li $z=\sqrt{3}-i$

1. postup: pomocí vzorce $\boxed{z=|z|\left(\frac{a}{|z|}+\frac{b}{|z|}i\right)}$

$$|z| = |\sqrt{3}-i| = \sqrt{3+1} = \sqrt{4} = 2$$

$$\sqrt{3}-i = 2\left(\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i\right) = 2\cdot\left(\cos\tfrac{\pi}{6} - i\sin\tfrac{\pi}{6}\right)$$

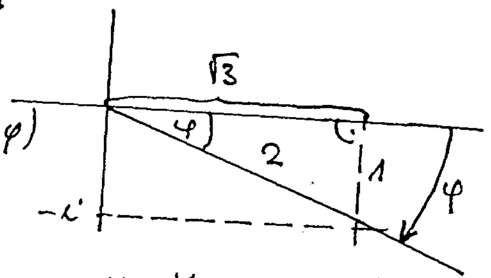
$$(\sqrt{3}-i)^8 = \left[2\left(\cos\tfrac{\pi}{6}\right) - i\sin\tfrac{\pi}{6}\right)\Big]^8 = 2^8\left(\cos\tfrac{8}{6}\pi - i\sin\tfrac{8}{6}\pi\right) =$$

$$= 2^8\left(\cos\tfrac{4}{3}\pi + i\sin\tfrac{4}{3}\pi\right) = 2^8\left[-\tfrac{1}{2} - \left(-\tfrac{\sqrt{3}}{2}\right)i\right] = 2^8\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) =$$

$$= 2^8\cdot\tfrac{1}{2}(-1+\sqrt{3}i) = \boxed{2^7(-1+\sqrt{3}i)}$$

2. postup: pomocí vzorce

$$[|z|(\cos\varphi + i\sin\varphi)]^n = |z|^n(\cos n\varphi + i\sin n\varphi)$$




$$(\sqrt{3}-i)^8 = \left[2\left(\cos\left(-\tfrac{\pi}{6}\right) + i\sin\left(-\tfrac{\pi}{6}\right)\right)\right]^8 =$$

$$= 2^8\left[\cos\left(-\tfrac{8}{6}\pi\right) + i\sin\left(-\tfrac{8}{6}\pi\right)\right] =$$

$$= 2^8\left[\cos\left(-\tfrac{4}{3}\pi\right) + i\sin\left(-\tfrac{4}{3}\pi\right)\right] =$$

$$= 2^8\left(\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\right) =$$

$$= 2^8\left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) = 2^8\cdot\tfrac{1}{2}(-1+i\sqrt{3}) = \boxed{2^7(-1+\sqrt{3}i)}$$

$\varphi$ určím pomocí tg

$$\operatorname{tg}\varphi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \varphi = \frac{\pi}{6}$$

Vzhledem k obrázku je to $-\frac{\pi}{6}$

$$2\pi - \tfrac{4}{3}\pi = \tfrac{2}{3}\pi$$

Příklad 13: Vypočítejte:

$$\left(\frac{1+i}{1-i}\right)^{25} = \left[\frac{(1+i)\cdot(1+i)}{(1-i)\cdot(1+i)}\right]^{25} = \left[\frac{(1+i)^2}{1^2-i^2}\right]^{25} = \left(\frac{1+2i+i^2}{1+1}\right)^{25} = \left(\frac{1+2i-1}{2}\right)^{25}$$

$$\left(\frac{2i}{2}\right)^{25} = i^{25} \text{ (podle vzorce na str. 7) } = i^{4\cdot 6+1} = i^{24+1} = \boxed{i}$$

Příklad 14: Vypočítejte:

a) $\left(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi\right)^{-50} = \dfrac{1}{\left(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi\right)^{50}} =$

$= \dfrac{1}{\cos\frac{50}{4}\pi + i\sin\frac{50}{4}\pi} = \dfrac{1}{\cos\frac{25}{2}\pi + i\sin\frac{25}{2}\pi} = \dfrac{1}{\cos \cancel{12}\frac{1}{2}\pi + i\sin\cancel{12}\frac{1}{2}\pi} =$

$= \dfrac{1}{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}} = \dfrac{1}{0 + i\cdot 1} = \dfrac{1}{i} = \dfrac{1\cdot(-i)}{i(-i)} = \dfrac{-i}{-i^2} = \dfrac{-i}{1} = \boxed{-i}$

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b) $\left(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi\right)^{-31} = \dfrac{1}{\left(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi\right)^{31}} = \dfrac{1}{\cos\frac{31}{6}\pi + i\sin\frac{31}{6}\pi} =$

$= \dfrac{1}{\cos 5\frac{1}{6}\pi + i\sin 5\frac{1}{6}\pi} = \dfrac{1}{\cos(\cancel{4\pi} + 1\frac{1}{6}\pi) + i\sin(\cancel{4\pi} + 1\frac{1}{6}\pi)} = \dfrac{1}{\cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi} =$

$=$ .. podle vzorce $\dfrac{1}{\cos\varphi + i\sin\varphi} = \cos(-\varphi) + i\sin(-\varphi)$ platí ...

$= \cos\left(-\frac{7}{6}\pi\right) + i\sin\left(-\frac{7}{6}\pi\right) = \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi =$ $\left(2\pi - \frac{7}{6}\pi = \frac{5}{6}\pi\right)$

$= \boxed{-\frac{\sqrt{2}}{2} + \frac{1}{2}i}$

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c) $\left(\dfrac{1+i}{1-i}\right)^{-25} = \left[\dfrac{(1+i)\cdot(1+i)}{(1-i)\cdot(1+i)}\right]^{-25} = \left[\dfrac{(1+i)^2}{1^2 - i^2}\right]^{-25} = \left[\dfrac{1+2i+i^2}{1+1}\right]^{-25} = \left(\dfrac{1+2i-1}{2}\right)^{-25} =$

$= \left(\dfrac{2i}{2}\right)^{-25} = (i)^{-25} = \dfrac{1}{i^{25}} = \dfrac{1}{i^{4\cdot 6+1}} = \dfrac{1}{i} =$ (Podle sh. 7: $25 = 4\cdot 6 + 1$)

$= \dfrac{1\cdot(-i)}{i\cdot(-i)} = \dfrac{-i}{-i^2} = \dfrac{-i}{-(-1)} = \dfrac{-i}{1} = \boxed{-i}$

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d) $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^5 = \left[1\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^5 =$ $|z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$= \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi = \boxed{-\frac{\sqrt{3}}{2} + \frac{1}{2}i}$

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