Binomické rovnice

Př. 3.8/84 nová uč. kompl. č. pro G

a) Řešte v C: $x^5-1-i\sqrt{3}=0$ ... upravíme na tvar

$x^5-(1+i\sqrt{3})=0$

$x^n-a=0$, popř. na tvar

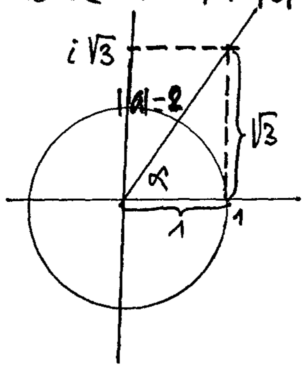
$x^n-|a|\cdot(\cos\alpha+i\sin\alpha)=0$, kde

$a\in C$, $n\in N-\{1\}$, čili $n>1$

Taková rovnice má v C $\underline{n}$ různých kořenů, a to

$$x_k=\sqrt[n]{|a|}\cdot\left(\cos\frac{\alpha+2k\pi}{n}+i\sin\frac{\alpha+2k\pi}{n}\right)$$

kde $k=0,1,2,3,\dots,(n-1)$, a je dané kompl. číslo



$|a|=\sqrt{1^2+(\sqrt{3})^2}=2$

$\cos\alpha=\frac{1}{2}$

$\Rightarrow\alpha=\frac{\pi}{3}\ (60^\circ)$

$\sin\alpha=\frac{\sqrt{3}}{2}$

$\Rightarrow\alpha=\frac{\pi}{3}$

$$x_k=\sqrt[5]{2}\cdot\left(\cos\frac{\frac{\pi}{3}+2k\pi}{5}+i\sin\frac{\frac{\pi}{3}+2k\pi}{5}\right)$$

kde $k=0,1,2,3,4$

Lze upravit, ale není to nutné.

$$x_k=\sqrt[5]{2}\cdot\left[\cos\left(\frac{\frac{\pi}{3}}{\frac{5}{1}}+\frac{2}{5}k\pi\right)+i\sin\left(\frac{\frac{\pi}{3}}{\frac{5}{1}}+\frac{2}{5}k\pi\right)\right]$$

$$x_k=\sqrt[5]{2}\cdot\left[\cos\left(\frac{\pi}{15}+\frac{2}{5}k\pi\right)+i\sin\left(\frac{\pi}{15}+\frac{2}{5}k\pi\right)\right]$$

dosaď postupně $k=0,1,2,3,4$

$$x_0=\sqrt[5]{2}\cdot\left(\cos\frac{\pi}{15}+i\sin\frac{\pi}{15}\right)$$

$$x_1=\sqrt[5]{2}\cdot\left[\cos\left(\frac{\pi}{15}+\frac{2}{5}\pi\right)+i\sin\left(\frac{\pi}{15}+\frac{2}{5}\pi\right)\right]$$

$$x_1=\sqrt[5]{2}\cdot\left(\cos\frac{7}{15}\pi+i\sin\frac{7}{15}\pi\right)$$

$$x_2=\sqrt[5]{2}\cdot\left[\cos\left(\frac{1}{15}\pi+\frac{2}{5}\cdot2\pi\right)+i\sin\left(\frac{1}{15}\pi+\frac{2}{5}\cdot2\cdot\pi\right)\right]$$

$$x_2=\sqrt[5]{2}\cdot\left(\cos\frac{13}{15}\pi+i\sin\frac{13}{15}\pi\right)$$

$$x_3=\sqrt[5]{2}\cdot\left(\cos\frac{19}{15}\pi+i\sin\frac{19}{15}\pi\right)$$

$$x_4=\sqrt[5]{2}\cdot\left(\cos\frac{5}{3}\pi+i\sin\frac{5}{3}\pi\right)$$

DŮLEŽITÁ POZNÁMKA:

Po výpočtu $x_0\dots\frac{\pi}{15}$, tak pak další úhly lze určit přičítáním $\frac{1}{5}$ velikosti 2π, tj. $\frac{1}{5}$ ze 2π = $\frac{1}{5}\cdot2\pi=\frac{2}{5}\pi$

Úhel pro ... protože $n=5$

x_1 je $\frac{1}{15}\pi+\frac{2}{5}\pi=\frac{7}{15}\pi$

x_2 je $\frac{7}{15}\pi+\frac{2}{5}\pi=\frac{13}{15}\pi$

x_3 je $\frac{13}{15}\pi+\frac{2}{5}\pi=\frac{19}{15}\pi$

x_4 je $\frac{19}{15}\pi+\frac{2}{5}\pi=\frac{5}{3}\pi$

Pro grafické vyjádření potřebuji:

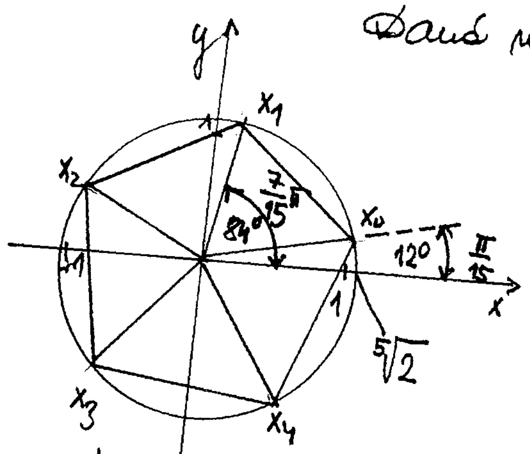
$\frac{1}{15}\pi=\frac{1}{15}\cdot180^\circ=12^\circ$

$\frac{7}{15}\pi=84^\circ\ \dots\ \frac{13}{15}\pi=156^\circ$

$\frac{19}{15}\pi=228^\circ\ \dots\ \frac{5}{3}\pi=300^\circ$

$\sqrt[5]{2}\doteq1{,}149$

Obrazy všech čtyř kořenů leží na kružnici o poloměrem $r\doteq1{,}149$



Daná rovnice má pět kořenů, a to:

$$\boxed{\begin{aligned}
x_0 &= \sqrt[5]{2}\cdot\left(\cos\frac{\pi}{15}+i\sin\frac{\pi}{15}\right)\\
x_1 &= \sqrt[5]{2}\cdot\left(\cos\frac{7}{15}\pi+i\sin\frac{7}{15}\pi\right)\\
x_2 &= \sqrt[5]{2}\cdot\left(\cos\frac{13}{15}\pi+i\sin\frac{13}{15}\pi\right)\\
x_3 &= \sqrt[5]{2}\cdot\left(\cos\frac{19}{15}\pi+i\sin\frac{19}{15}\pi\right)\\
x_4 &= \sqrt[5]{2}\cdot\left(\cos\frac{5}{3}\pi+i\sin\frac{5}{3}\pi\right)
\end{aligned}}$$

Obrazy těchto kořenů tvoří vrcholy pravidelného pětiúhelníku.

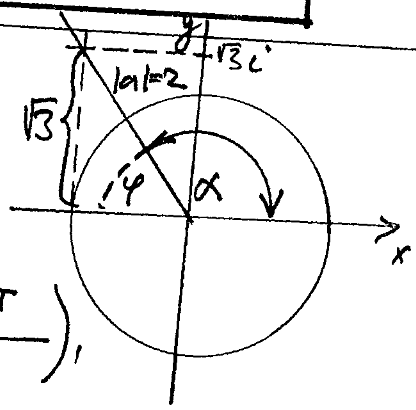
b) $x^5+1-i\sqrt{3}=0$

$x^5-(-1+i\sqrt{3})=0$

$|a|=|-1+i\sqrt{3}|=$

$=\sqrt{(-1)^2+(\sqrt{3})^2}=2$

$|a|=2$



$$x_k=\sqrt[5]{2}\cdot\left(\cos\frac{\frac{2}{3}\pi+2k\pi}{5}+i\sin\frac{\frac{2}{3}\pi+2k\pi}{5}\right),$$

kde $k=0,1,2,3,4$

$$x_0=\sqrt[5]{2}\cdot\left(\cos\frac{\frac{2\pi}{3}}{\frac{5}{1}}+i\sin\frac{\frac{2\pi}{3}}{\frac{5}{1}}\right)\ \ldots\ \frac{2}{15}\pi=\frac{2}{15}\cdot 180°=24°$$

$$\boxed{\begin{aligned}
x_0 &= \sqrt[5]{2}\cdot\left(\cos\frac{2}{15}\pi+i\sin\frac{2}{15}\pi\right)\ \to (24°)\\
x_1 &= \sqrt[5]{2}\cdot\left(\cos\frac{8}{15}\pi+i\sin\frac{8}{15}\pi\right)\\
x_2 &= \sqrt[5]{2}\cdot\left(\cos\frac{14}{15}\pi+i\sin\frac{14}{15}\pi\right)\\
x_3 &= \sqrt[5]{2}\cdot\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)\\
x_4 &= \sqrt[5]{2}\cdot\left(\cos\frac{26}{15}\pi+i\sin\frac{26}{15}\pi\right)
\end{aligned}}$$

... opět přičtu $\frac{1}{5}$ velikosti $2\pi=\frac{2}{5}\pi$

Pro x_1 je $\alpha_1=\frac{2}{15}\pi+\frac{2}{5}\pi=\frac{8}{15}\pi\ (96°)$

Pro x_2 je $\alpha_2=\frac{8}{15}\pi+\frac{2}{5}\pi=\frac{14}{15}\pi\ (168°)$

Pro x_3 je $\frac{14}{15}\pi+\frac{2}{5}\pi=\frac{4}{3}\pi\ (240°)$

Pro x_4: $\frac{4}{3}\pi+\frac{2}{5}\pi=\frac{26}{15}\pi\ (312°)$

Výsledky v stupních lze psát též:

$x_0=\sqrt[5]{2}\cdot(\cos 24°+i\sin 24°)$

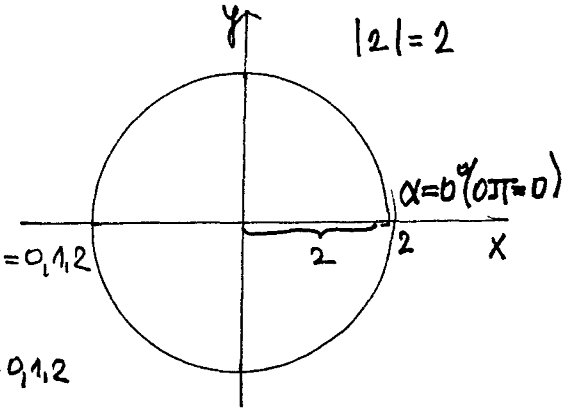
$x_1=\sqrt[5]{2}\cdot(\cos 96°+i\sin 96°)$

$x_2=\sqrt[5]{2}\cdot(\cos 168°+i\sin 168°)$

$x_3=\sqrt[5]{2}\cdot(\cos 240°+i\sin 240°)$

$x_4=\sqrt[5]{2}\cdot(\cos 312°+i\sin 312°)$

Př. z nedodělaného zadání: Řešte rovnici $x^3-2=0$



$x^3-2=0$

$x^3-(+2)=0$

$x_k=\sqrt[3]{2}\cdot\left(\cos\frac{0+2k\pi}{3}+i\sin\frac{0+2k\pi}{3}\right)\ldots k=0,1,2$

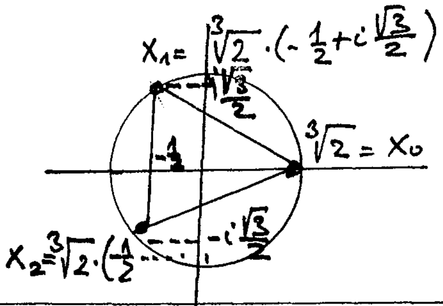
$x_k=\sqrt[3]{2}\cdot\left(\cos\frac{2k\pi}{3}+i\sin\frac{2k\pi}{3}\right)$, kde $k=0,1,2$

$x_0=\sqrt[3]{2}\cdot(\cos 0+i\sin 0)\quad\ldots x_0=\sqrt[3]{2}\cdot(1+0)\ \ldots\ x_0=\sqrt[3]{2}\quad\ldots x_0=\sqrt[3]{2}$

$x_1=\sqrt[3]{2}\cdot\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)\ldots x_1=\sqrt[3]{2}\cdot\left(\cos 120^\circ+i\sin 120^\circ\right)\ldots x_1=\sqrt[3]{2}\cdot\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$x_2=\sqrt[3]{2}\cdot\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)\ldots\ x_2=\sqrt[3]{2}\cdot\left(\cos 240^\circ+i\sin 240^\circ\right)\ldots x_2=\sqrt[3]{2}\cdot\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$

Obraz všech tří kořenů leží
na kružnici s poloměrem
$r=\sqrt[3]{2}\doteq 1{,}26$



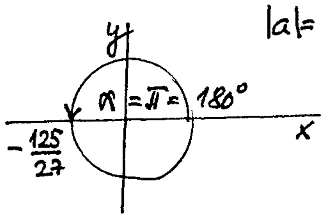
Př. 1/151 – sbírka úč. M IV. pro G : Řešte rovnici $27x^3+125=0$

$27x^3+125=0 \quad |:27$

$x^3+\frac{125}{27}=0$

$x^3-\underbrace{\left(-\frac{125}{27}\right)}_{a}=0$

$|a|=\left|-\frac{125}{27}\right|=\frac{125}{27}$



$\sqrt[3]{\frac{125}{27}}=\frac{5}{3}$

$x_k=\sqrt[3]{\frac{125}{27}}\cdot\left(\cos\frac{\pi+2k\pi}{3}+i\sin\frac{\pi+2k\pi}{3}\right)$

$x_k=\frac{5}{3}\cdot\left[\cos\left(\frac{\pi}{3}+\frac{2}{3}k\pi\right)+i\sin\left(\frac{\pi}{3}+\frac{2}{3}k\pi\right)\right.$, kde $k=0,1,2$

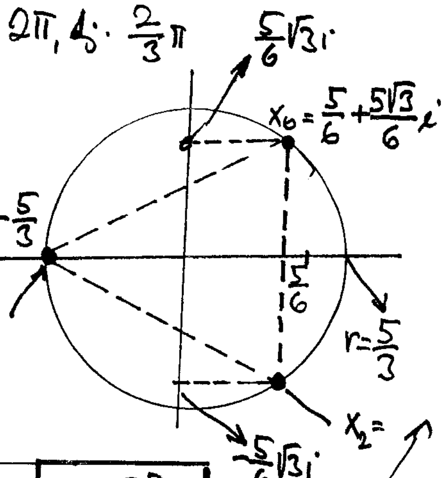
$\boxed{x_0=\frac{5}{3}\cdot\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}=\frac{5}{3}\left(\frac{1}{2}\,i\,\frac{\sqrt{3}}{2}\right)=\boxed{\frac{5}{6}+\frac{5\sqrt{3}}{6}i}$

$x_0 = \frac{5}{3}\cdot(\cos 60° + i \sin 60°)$

Po výpočtu $\frac{\pi}{3}$ pro x_0 pak přičteme $\frac{1}{3}$ ze 2π, tj. $\frac{2}{3}\pi$

Pro x_1 to bude tedy $\frac{\pi}{3} + \frac{2}{3}\pi = \frac{3}{3}\pi = \pi$

Pro x_2 " " " $\pi + \frac{2}{3}\pi = \frac{5}{3}\pi$



$\boxed{x_1 = \frac{5}{3}\left(\cos\pi + i \sin\pi\right)} = \frac{5}{3}\cdot(-1 + 0i) = \boxed{-\frac{5}{3}}$

$x_1 = \frac{5}{3}\left(\cos 180° + i \sin 180°\right)$

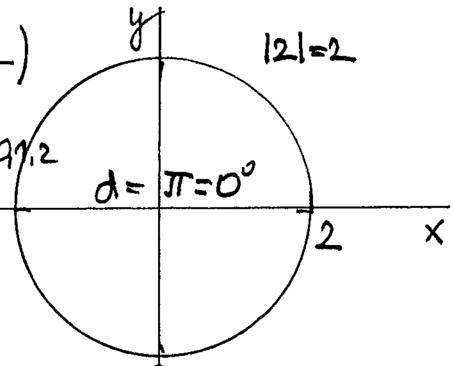
$\boxed{x_2 = \frac{5}{3}\left(\cos\frac{5}{3}\pi + i \sin\frac{5}{3}\pi\right)} = \frac{5}{3}\cdot\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \boxed{\frac{5}{6} - \frac{5\sqrt{3}}{6}i}$

Rovnice má řešení: $x_0 = \frac{5}{3}\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$

$x_1 = \frac{5}{3}\left(\cos\pi + i \sin\pi\right)$

$x_2 = \frac{5}{3}\left(\cos\frac{5}{3}\pi + i \sin\frac{5}{3}\pi\right)$

Př. 1/70 z Přehledu užité matematiky. Vypočítej $x^3 - 2 = 0$



$x^3 - 2 = 0 \qquad x_k = \sqrt[3]{2}\cdot\left(\cos\frac{0+2k\pi}{3} + i \sin\frac{0+2k\pi}{3}\right)$

$x^3 - \underbrace{(+2)}_{a} = 0 \qquad x_k = \sqrt[3]{2}\cdot\left(\cos\frac{2}{3}k\pi + i \sin\frac{2}{3}k\pi\right);\ k = 0,1,2$

$\boxed{x_0 = \sqrt[3]{2}\cdot(\cos 0 + i \sin 0)} = \sqrt[3]{2}\cdot(1 + 0i) = \boxed{\sqrt[3]{2}} \doteq 1{,}26$

Přičítáme $\frac{1}{3}$ ze $2\pi = \frac{2}{3}\pi$

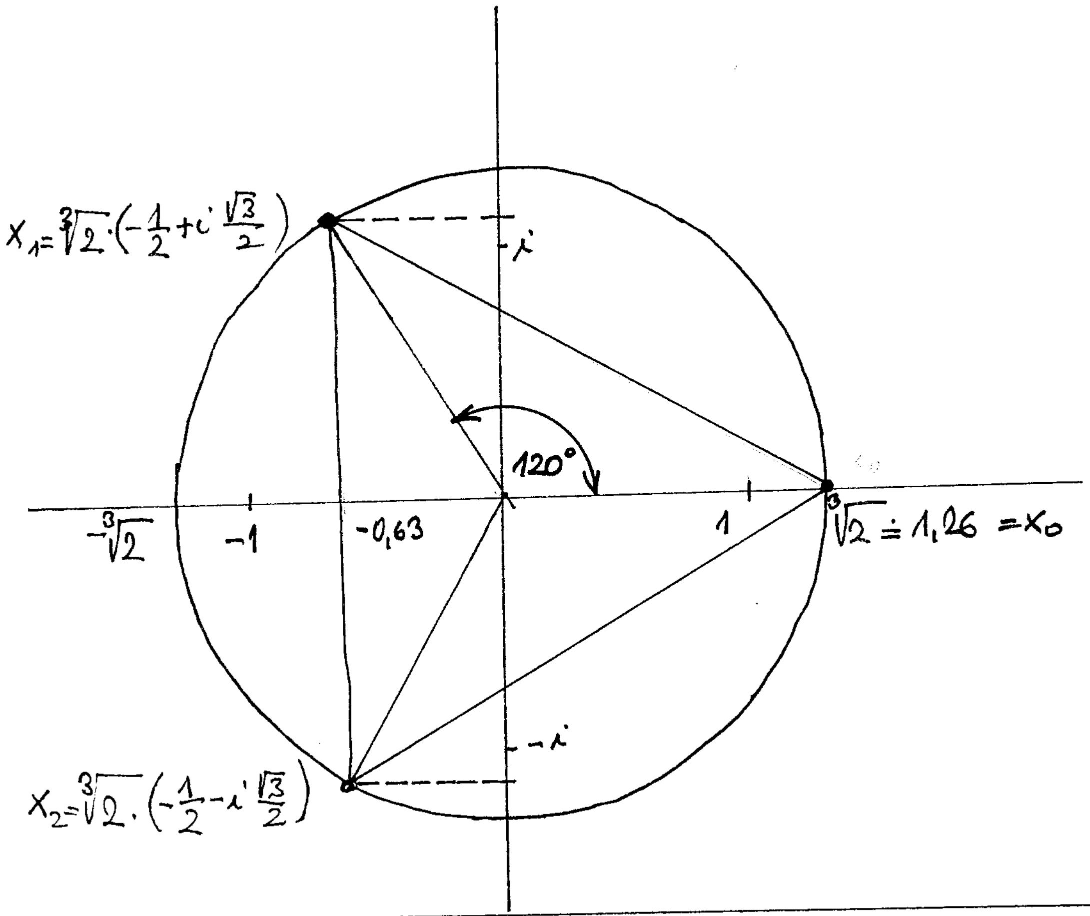
$\boxed{x_1 = \sqrt[3]{2}\cdot\left(\cos\frac{2}{3}\pi + i \sin\frac{2}{3}\pi\right)} = \boxed{\sqrt[3]{2}\cdot\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)} = -\frac{\sqrt[3]{2}}{2} + \frac{\sqrt[3]{2}\cdot\sqrt[2]{3}}{2}i =$

$= -\frac{\sqrt[3]{2}}{2} + i\,\frac{\sqrt[3]{2}\cdot\sqrt[2]{3}}{2}\,\cancel{i}$

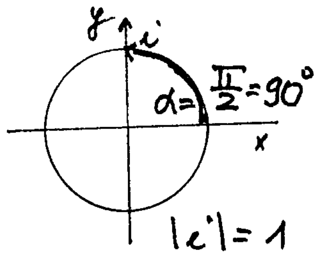
$\doteq -0{,}63 + 1{,}1i$

$\boxed{x_2 = \sqrt[3]{2}\cdot\left(\cos\frac{4}{3}\pi + i \sin\frac{4}{3}\pi\right)} = \boxed{\sqrt[3]{2}\cdot\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)} = -\frac{\sqrt[3]{2}}{2} - \frac{\sqrt[6]{2^5}}{2}$

$\doteq -0{,}63 - 1{,}1i$



Př. z posledního zadání: Řeš rovnici $z^2 = i$



$z^2 = i \qquad z_k = \sqrt{1}\cdot\left(\cos\frac{\frac{\pi}{2}+2k\pi}{2} + i\sin\frac{\frac{\pi}{2}+2k\pi}{2}\right)$

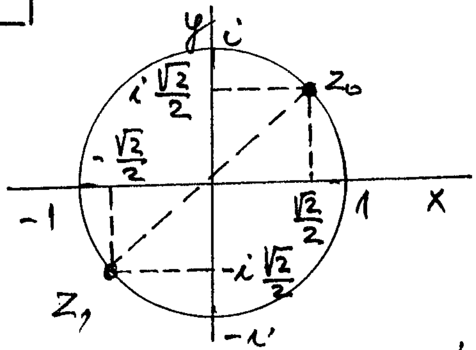
$z^2 - i = 0$

$z^2 - (+i) = 0 \qquad z_0 = 1\cdot\left(\cos\frac{\frac{\pi}{2}}{\frac{2}{1}} + i\sin\frac{\frac{\pi}{2}}{\frac{2}{1}}\right)$

$\boxed{z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}} = \boxed{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}$ Přičteme $\frac{2\pi}{2} = \pi$

$\boxed{z_1 = \cos\left(\frac{\pi}{4}+\pi\right) + i\sin\left(\frac{\pi}{4}+\pi\right)}$

$\boxed{z_1 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}$



Šlo tedy o kvadratickou rovnici s komplexními koeficienty. Ty budeme nyní řešit v samostatném článku.